Ma 221 - Exam II review

Second Order Differential Equations

Form of general solution

$$y_h = c_1 y_1 + c_2 y_2$$

where y_1 and y_2 are linearly independent solutions of the homogeneous equation and

$$y = y_h + y_p$$

where y_p is a [particular] so;lution of the non-homogeneous equation.

Wronskian - provides a test for linear independence of solutions

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

$W[y_1, y_2] \neq 0$ for linearlly independent solutions of the homogeneous d.e.

Homogeneous D.E.

Constant coefficients -

$$ay'' + by' + cy = 0$$

Solve auxiliary (characteristic) equation -

$$p(r) = ar^2 + br + c = 0$$

2 real roots

$$r = r_1, r_2$$

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

repeated real roots

$$r = r_1$$

$$y_h = (c_1 + c_2 x) e^{r_1 x}$$

2 complex roots

$$r = \alpha \pm i\beta$$

$$y_h = (c_1 \cos \beta x + c_2 \sin \beta x) e^{\alpha x}$$

Cauchy-Euler D.E. -

$$ax^2y'' + bxy' + cy = 0$$

Solve auxiliary (indicial) equation -

$$am^2 + (b - a)m + c = 0$$

Or one can divide by a and put the equation in the form

$$x^2y'' + pxy' + qy = 0$$

Then the auxiliary (indicial) equation that one must solve for m is

$$m^2 + (p - 1)m + q = 0$$

2 real roots

$$m = m_1, m_2$$

$$y_h = c_1x^{m_1} + c_2x^{m_2}$$

repeated real roots

$$m = m_1$$

$$y_h = (c_1 + c_2 \ln x)x^{m_1}$$

2 complex roots

$$m = \alpha \pm i\beta$$

$$y_h = (c_1 \cos \beta \ln x + c_2 \sin \beta \ln x)x^{\alpha}$$

Non-homogeneous D.E.

Undetermined coefficients

Constant coefficient d.e.

$$ay'' + by' + cy = f(x)$$

$$f(x) = Ae^{\alpha x}$$

$$f(x) = (A\cos\beta x + B\sin\beta x)e^{\alpha x}$$

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

and products of above

Be careful if $f(x)$ is a solution of the homogeneous equation.

In the case with $f(x) = Ae^{\alpha x}$, the following formulae give the solution.

$$y(x) = \begin{cases} \frac{A}{p(\alpha)}e^{\alpha x} & p(\alpha) \neq 0 \\ \frac{A}{p'(\alpha)}xe^{\alpha x} & p(\alpha) = 0, p'(\alpha) \neq 0 \\ \frac{A}{p''(\alpha)}x^2e^{\alpha x} & p(\alpha) = p'(\alpha) = 0 \end{cases}$$

Cauchy-Euler D.E.

$$ax^2y'' + bxy' + cy = f(x)$$

$$f(x) = Ax^p$$

If $f(x)$ is a solution of the homogeneous equation, use variation of parameters.

Variation of parameters

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

$$y = v_1y_1 + v_2y_2$$

$$y_1v_1' + y_2v_2' = 0$$

$$y_1'v_1' + y_2'v_2' = \frac{f(x)}{a(x)}$$

$$v_1' = \frac{-f(x)y_2}{a(x)(y_1y_2' - y_2y_1')}$$

$$v_2' = \frac{f(x)y_1}{a(x)(y_1y_2' - y_2y_1')}$$