Ma 221 - Course Overview

First Order Differential Equations

Separable Equations

$$\frac{dy}{dx} = g(x)p(y)$$
$$h(y)dy = g(x)dx$$
$$\int h(y)dy = \int g(x)dx$$

Linear Equations

$$\frac{dy}{dx} + p(x)y = q(x)$$

Integrating Factor

$$IF = e^{\int p(x)dx}$$

$$e^{\int p(x)dx}\left(\frac{dy}{dx} + p(x)y\right) = e^{\int p(x)dx}[q(x)]$$
$$\frac{d}{dx}\left(e^{\int p(x)dx}y\right) = e^{\int p(x)dx}[q(x)]$$
$$e^{\int p(x)dx}y = \int\left(e^{\int p(x)dx}[q(x)]\right)dx$$
$$y = e^{-\int p(x)dx}\int\left(e^{\int p(x)dx}[q(x)]\right)dx$$

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0$$

Test for exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

When test is passed, there is $F(x,y)$ such that

$$M = \frac{\partial F}{\partial x} \qquad and \quad N = \frac{\partial F}{\partial y}$$

Find $F(x,y)$. Solution is

$$F(x,y) = c$$

Bernoulli D.E.

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

Substitution

$$z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

yields a linear d.e. in z.

In all cases, the arbitrary constant resulting from integration is used to satisfy any initial condition.

Second Order Differential Equations

Form of general solution

$$y_h = c_1 y_1 + c_2 y_2$$

where y_1 and y_2 are linearly independent solutions of the homogeneous equation and

$$y = y_h + y_p$$

where y_p is a {particular] solution fo the non-homogeneous equation.

Wronskian - provides a test for linear independence of solutions

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Homogeneous D.E.

Constant coefficients -

$$ay'' + by' + cy = 0$$

Solve auxiliary (characteristic) equation -

$$p(r) = ar^2 + br + c = 0$$

2 real roots

$$r = r_1, r_2$$

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

repeated real roots

$$r = r_1$$

$$y_h = (c_1 + c_2 x)e^{r_1 x}$$

2 complex roots

$$r = \alpha \pm i\beta$$
$$y_h = (c_1 \cos \beta x + c_2 \sin \beta x)e^{\alpha x}$$

Cauchy-Euler D.E. -

$$ax^2y'' + bxy' + cy = 0$$

Solve auxiliary (indicial) equation -

$$am^2 + (b-a)m + c = 0$$

2 real roots

$$m = m_1, m_2$$
$$y_h = c_1 x^{m_1} + c_2 x^{m_2}$$

repeated real roots

$$m = m_1$$
$$y_h = (c_1 + c_2 \ln x)x^{m_1}$$

2 complex roots

$$m = \alpha \pm i\beta$$
$$y_h = [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]x^{\alpha}$$

Non-homogeneous D.E.

Undetermined coefficients

Constant coefficient d.e.

$$ay'' + by' + cy = f(x)$$
$$f(x) = ce^{\alpha x}$$
$$f(x) = (A \cos \beta x + B \sin \beta x)e^{\alpha x}$$
$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

products of above

Some simple formulae for the case $f(x) = ce^{\alpha x}$

$$y_p = \frac{ce^{\alpha x}}{p(\alpha)} \qquad p(\alpha) \neq 0$$
$$y_p = \frac{cxe^{\alpha x}}{p'(\alpha)} \qquad p(\alpha) = 0 \; and \; p'(\alpha) \neq 0$$
$$y_p = \frac{cx^2e^{\alpha x}}{p''(\alpha)} \qquad p(\alpha) = p'(\alpha) = 0$$

Be careful if $f(x)$ is a solution of the homogeneous equation.

Cauchy-Euler D.E.

$$ax^2y'' + bxy' + cy = f(x)$$
$$f(x) = kx^p$$

Be very careful if $f(x)$ is a solution of the homogeneous equation or use variation of parameters.

Variation of parameters

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$
$$y = v_1y_1 + v_2y_2$$
$$y_1v_1' + y_2v_2' = 0$$
$$y_1'v_1' + y_2'v_2' = \frac{f(x)}{a(x)}$$
$$v_1' = \frac{-\frac{f(x)}{a(x)}y_2}{y_1y_2' - y_2y_1'}$$
$$v_2' = \frac{\frac{f(x)}{a(x)}y_1}{y_1y_2' - y_2y_1'}$$

Laplace Transforms

Definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$$
$$= F(s) = \hat{f}(s)$$

Calculate Laplace Transform from definition

Properties

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y\} - y(0)$$
$$\mathcal{L}\left\{y''(t)\right\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

Inverse Laplace Transform

Partial Fractions

Use of Laplace transform for a solution of initial value problems

Power series solution of ordinary differential equations

Recurrence relation

Boundary Value Problems

Eigenvalues and eigenfunctions

$$DE: \quad L[y] + \lambda y = 0$$
$$BC: \quad \alpha_1 y(a) + \beta_1 y'(a) = 0$$
$$BC: \quad \alpha_2 y(b) + \beta_2 y'(b) = 0$$

Three cases to be examined (discriminant positive, zero or negative)

Fourier Series

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left[\left(\frac{n\pi}{L}\right)x\right]dx$$

Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left[\left(\frac{n\pi}{L}\right)x\right]dx$$

Convergence

$f(x)$ when f is continuous at x and $0 < x < L$

$\frac{f(x-)+f(x+)}{2}$ Average value at jumps

Extension (odd for sine series, even for cosine series) $-L < x < L$

Periodic extension with period $2L$

Partial Differential Equations

Separation of Variables

Eigenvalue problem from boundary conditions

Fourier expansion using eigenfunctions