Ma 221 Homework Solutions Spring 2015
Due January 22, 2015

1.2 p.13-14 #2, 4, 5, 6, 7, 8, 10, 11, 17, 20b, 21b, 22a,b

2.

(a) Show that $\Phi(x) = x^2$ is an explicit solution to $x\frac{dy}{dx} = 2y$ on the interval $(-\infty, \infty)$.

Differentiating $\Phi(x)$ gives:

$\Phi'(x) = 2x$

Substituting Φ and Φ' for y and y':

$x\frac{dy}{dx} = 2y$

$xy' = 2y$

$x(2x) = 2(x^2)$

$2x^2 = 2x^2$

This identity is true on $(-\infty, \infty)$ and therefore $\Phi(x)$ is an explicit solution on $(-\infty, \infty)$.

(b) Show that $\Phi(x) = e^x - x$ is an explicit solution to $\frac{dy}{dx} + y^2 = e^{2x} + (1-2x)e^x + x^2 - 1$ on the interval $(-\infty, \infty)$.

Differentiating $\Phi(x)$ gives:

$\frac{d\Phi}{dx} = \frac{d}{dx}(e^x - x) = e^x - 1$

Substituting Φ and Φ' for y and y':

$$\frac{d\Phi}{dx} + \Phi(x)^2 = (e^x - 1) + (e^x - x)^2 = (e^x - 1) + (e^{2x} - 2xe^x + x^2) = e^{2x} + (1-2x)e^x + x^2 - 1$$

$$e^{2x} + (1-2x)e^x + x^2 - 1 = e^{2x} + (1-2x)e^x + x^2 - 1$$

Since both sides of the equation are equal, $\Phi(x)$ is an explicit solution on $(-\infty, \infty)$.

(c) Show that $\Phi(x) = x^2 - x^{-1}$ is an explicit solution to $x^2\frac{d^2y}{dx^2} = 2y$ on the interval $(0, \infty)$.

Differentiating $\Phi(x)$ twice gives:

$\frac{d\Phi}{dx} = \frac{d}{dx}(x^2 - x^{-1}) = 2x - (-1)x^{-2} = 2x + x^{-2}$

$\frac{d^2\Phi}{dx^2} = \frac{d}{dx}(\frac{d\Phi}{dx}) = \frac{d}{dx}(2x + x^{-2}) = 2 + (-2)x^{-3} = 2(1 - x^{-3})$

Therefore

$x^2\frac{d^2\Phi}{dx^2} = x^2.2(1 - x^{-3}) = 2(x^2 - x^{-1}) = 2\Phi(x)$

Both sides of the equation are equal therefore, $\Phi(x)$ is an explicit solution to the differential equation $x^2y'' = 2y$ on any interval that does not contain the point $x = 0$, in particular, on $(0, \infty)$.

4. Determine whether the function $y = \sin x + x^2$ is a solution to the differential equation $\frac{d^2y}{dx^2} + y = x^2 + 2$.

Since $y = \sin x + x^2$, we have $y' = \cos x + 2x$ and $y'' = -\sin x + 2$. These functions are defined on $(-\infty, \infty)$ Substituting these expressions into the differential equation $y'' + y = x^2 + 2$ gives $y'' + y = -\sin x + 2 + \sin x + x^2 = 2 + x^2 = x^2 + 2$ for all x in $(-\infty, \infty)$.

Therefore, $y = \sin x + x^2$ is a solution to the differential equation on the interval $(-\infty, \infty)$.

5. Determine whether the given funtion is a solution to the given differential equation:

$x = \cos 2t$ $\qquad\qquad$ $\frac{dx}{dt} + tx = \sin 2t$

Differentiating $x(t) = \cos 2t$, we get:

$\frac{dx}{dt} = \frac{d}{dt}(\cos 2t) = (-\sin 2t)(2) = -2\sin 2t$

$\frac{dx}{dt} + tx = -2\sin 2t + t\cos 2t$

which is not equal to $\sin 2t$. Therefore, $x(t)$ is not a solution to the given differential equation.

6. Determine whether the given funtion is a solution to the given differential equation:

$\theta = 2e^{3t} - e^{2t}$ $\qquad\qquad$ $\frac{d^2\theta}{dt^2} - \theta\frac{d\theta}{dt} + 3\theta = -2e^{2t}$

Differentiating $\theta = 2e^{3t} - e^{2t}$, we get

$\theta' = 6e^{3t} - 2e^{2t}$ $\qquad\qquad$ $\theta'' = 18e^{3t} - 4e^{2t}$

$\frac{d^2\theta}{dt^2} - \theta\frac{d\theta}{dt} + 3\theta = 18e^{3t} - 4e^{2t} - \theta(6e^{3t} - 2e^{2t}) + 3(2e^{3t} - e^{2t}) = 24e^{3t} - 7e^{2t} + 2\theta e^{2t} - 6\theta e^{3t}$ $\neq -2e^{2t}$.

Therefore, θ is not a solution to the given differential equation.

7. Determine whether the given funtion is a solution to the given differential equation:

$y = 3\sin 2x + e^{-x}$ $\qquad\qquad$ $y'' + 4y = 5e^{-x}$

Differentiating $y(x) = 3\sin 2x + e^{-x}$ twice, we get:

$\frac{dy}{dx} = \frac{d}{dx}(3\sin 2x + e^{-x}) = 6\cos 2x - e^{-x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(6\cos 2x - e^{-x}) = -12\sin 2x + e^{-x}$

$y'' + 4y = -12\sin 2x + e^{-x} + 4(3\sin 2x + e^{-x}) = -12\sin 2x + e^{-x} + 12\sin 2x + 4e^{-x} = 5e^{-x}$

which is equal to $5e^{-x}$. Therefore, $y(t)$ is a solution to the given differential equation.

8. Determine whether the given funtion is a solution to the given differential equation:

$$y = e^{2x} - 3e^{-x} \qquad\qquad \frac{d^2y}{dy^2} - \frac{dy}{dx} - 2y = 0$$

Differentiating $y = e^{2x} - 3e^{-x}$, we get
$y' = 2e^{2x} + 3e^{-x}$; $y'' = 4e^{2x} - 3e^{-x}$
$\frac{d^2y}{dy^2} - \frac{dy}{dx} - 2y = 4e^{2x} - 3e^{-x} - 2e^{2x} - 3e^{-x} - 2(e^{2x} - 3e^{-x}) = 0.$
Therefore, $y = e^{2x} - 3e^{-x}$is an explicit solution to the given differential equation.

In problems 10 and 11, determine whether the given relation is an implicit solution to the given differential equation.

10.) $x^2 + y^2 = 4$, $\frac{dy}{dx} = \frac{x}{y}$

Implicitly differentiate to get:
$2x + 2yy' = 0$
Solve to get:
$y' = -\frac{x}{y}$
Therefore, $x^2 + y^2 = 4$ is not a solution.

11.) $e^{xy} + y = x - 1$, $\frac{dy}{dx} = (e^{-xy} - y)/(e^{-xy} + x)$

Differentiate implicitly to get:
$\frac{d}{dx}(e^{xy} + y) = \frac{d}{dx}(x - 1)$
$e^{xy}\frac{d}{dx}(xy) + \frac{dy}{dx} = 1$
$e^{xy}(y + x\frac{dy}{dx}) + \frac{dy}{dx} = 1$
$\frac{dy}{dx}(e^{xy}x + 1) = 1 - e^{xy}y$
$xe^{xy}y' + ye^{xy} + y' = 1$
Now, solve for y':
$y' = (1 - ye^{xy})/(xe^{xy} + 1)$
Multiply the expression by e^{xy}/e^{xy} :
$y' = (e^{-xy} - y)/(e^{-xy} + x)$
Thus, it is a solution.

17.) Show that $\varphi(x) = Ce^{3x} + 1$ is a solution to $\frac{dy}{dx} - 3y = -3$ for any choice of the constant C.

Differentiate $\varphi(x)$ to get:
$\varphi'(x) = 3Ce^{3x}$
Now substitute y for φ and y' for φ' so that:
$y' - 3y = 3Ce^{3x} - 3(Ce^{3x} + 1) = -3$
Simplify to get:
$-1 = -1$
Which is true for any constant C.

20b) Determine for which values of m the function $\varphi(x) = e^{mx}$ is a solution to

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$\varphi(x) = e^{mx}$
$\varphi'(x) = me^{mx}$
$\varphi''(x) = m^2e^{mx}$

$\varphi'''(x) = m^3 e^{mx}$
Now, substitute into the DE to get:
$m^3 e^{mx} + 3(m^2 e^{mx}) + 2(me^{mx}) = 0$
$m^3 + 3m^2 + 2m = 0$
$m(m^2 + 3m + 2) = 0$
$m(m+2)(m+1) = 0$
$m = 0, -1, -2$

21b) Determine for which values of m the function $\varphi(x) = x^m$ is a solution to

$$x^2 y'' - xy' - 5y = 0$$
$$\varphi'(x) = mx^{m-1}$$
$$\varphi''(x) = m(m-1)x^{m-2}$$

Substituting into the DE we have

$$x^2[m(m-1)x^{m-2}] - x[mx^{m-1}] - 5x^m = 0$$

or

$$(m^2 - 2m - 5)x^m = 0$$

Hence

$$m = \frac{2 \pm \sqrt{4+20}}{2} = 1 \pm \sqrt{6}$$

22.) Verifty that the function $\varphi(x) = c_1 e^x + c_2 e^{-2x}$ is a solution to the linear equation
$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$
for any choice of c_1 and c_2 so that each of the following initial conditions is satisfied:
(a.) $y(0) = 2, \quad y'(0) = 1$
(b.) $y(1) = 1, \quad y'(1) = 0$

$\varphi(x) = c_1 e^x + c_2 e^{-2x}$
$\varphi'(x) = c_1 e^x - 2c_2 e^{-2x}$
$\varphi''(x) = c_1 e^x + 4c_2 e^{-2x}$
Substituting into the DE we get:
$c_1 e^x + 4c_2 e^{-2x} + c_1 e^x - 2c_2 e^{-2x} - 2(c_1 e^x + c_2 e^{-2x}) = 0$
Which simplifies to

$$0 = 0$$

and is a solution.
(a.) Plugging in the initial conditions we get:
$\varphi(0) = c_1 + c_2 = 2$
$\varphi'(0) = c_1 - 2c_2 = 1$
Solving the system, we get $c_1 = \frac{5}{3}$ and $c_2 = \frac{1}{3}$.
(b.) Plugging in the initial conditions we get:
$\varphi(1) = c_1 e + c_2 e^{-2} = 1$
$\varphi'(1) = c_1 e - 2c_2 e^{-2} = 0$

Solving the system, we get $c_1 = \frac{2}{3e}$ and $c_2 = \frac{1}{3e^{-2}}$.