MA 221 Homework Solutions
Due February 12, 2015

4.2 p. 165 - 166 #17, 26, <u>27</u> **29**
4.3, pg. 173 # 2, <u>4</u>, 6, <u>8</u>, <u>17</u>, <u>27</u>, <u>29b</u>
(Underlined problems are to be handed in)
Section 4.2
For problem 17, solve the intial value problem.
17.) $z'' - 2z' - 2z = 0 \qquad z(0) = 0 \qquad z'(0) = 3$
The auxiliary equation for this problem, $r^2 - 2r - 2 = 0$, has roots $r = 1 \pm \sqrt{3}$.
Thus, a general solution is given by $z(t) = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}$
Differentiating, we find that $z'(t) = c_1(1+\sqrt{3})e^{(1+\sqrt{3})t} + c_2(1-\sqrt{3})e^{(1-\sqrt{3})t}$.
Substituting $z(t)$ and $z'(t)$ into the initial condition yields the system
$z(0) = c_1 + c_2 = 0 \qquad \Rightarrow$
$z'(0) = c_1(1+\sqrt{3}) + c_2(1-\sqrt{3}) = \sqrt{3}(c_1 - c_2) = 3$
$\Rightarrow c_1 = \sqrt{3}/2$
$c_2 = -\sqrt{3}/2$
Thus, the solution satisfying the given initial conditions is
$z(t) = \sqrt{3}/2e^{(1+\sqrt{3})t} - \sqrt{3}/2e^{(1-\sqrt{3})t} = \sqrt{3}/2(e^{(1+\sqrt{3})t} - e^{(1-\sqrt{3})t})$

26.) Boundary Value Problems
Given that every solution to $y'' + y = 0$ is of the form $y(x) = c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are arbitrary constants, show that
(a) there is a unique solution to $y'' + y = 0$ that satisfies the boundary conditions $y(0) = 2$ and $y(\pi/2) = 0$.
(b) there is no solution to $y'' + y = 0$ that satisfies $y(0) = 2$ and $y(\pi) = 0$
(c) there are infinitely many solutions to $y'' + y = 0$ that satisfy $y(0) = 2$ and $y(\pi) = -2$

(a) Plugging $y(0) = 2$ into $y(x)$
$2 = C_1 \cos 0 + C_2 \sin 0$
Therefore
$C_1 = 2$
Plugging $y(\pi/2) = 0$ into $y(x)$
$0 = 2\cos\frac{\pi}{2} + C_2 \sin\frac{\pi}{2}$
Therefore
$C_2 = 0$
The solution with these conditions is
$y(x) = 2\cos x$

(b) Plugging $y(0) = 2$ into $y(x)$
$2 = C_1 \cos 0 + C_2 \sin 0$
Therefore

$C_1 = 2$
Plugging $y(\pi) = 0$ into $y(x)$
$0 = 2\cos\pi + C_2\sin\pi$
The term with C_2 drops out leaving the equation
$0 = -2$
This is not true, and thus there is no solution that satisfies these boundary conditions.

(c) Plugging $y(0) = 2$ into $y(x)$
$2 = C_1\cos 0 + C_2\sin 0$
Therefore
$C_1 = 2$
Plugging $y(\pi) = -2$ into $y(x)$
$-2 = 2\cos\pi + C_2\sin\pi$
The term with C_2 drops out leaving the equation
$-2 = -2$
This is true, and thus there are infinitely many solutions with these boundary conditions.

27.) Determine whether the functions $y_1(t) = \cos t\sin t$; $y_2(t) = \sin 2t$ are linearly independent on the interval $(0,1)$.

Since $\sin 2t = 2\cos t\sin t$, we have that $y_1 = 2y_2$ and the functions are not LI.

29.) Determine whether the functions $y_1(t) = te^{2t}$; $y_2(t) = e^{2t}$ are linearly independent on the interval $(0,1)$.
(For this problem we use the Wronskian.)
These two functions are solutions of the DE
$y'' - 4y' + 4y = 0$
since this equation has the characteristic equation
$r^2 - 4r + 4 = (r-2)^2 = 0$
and has the repeated root $r = 2$. Thus we may check to Wronskian. to see if it is zero or not. Now
$$W[te^{2t}, e^{2t}] = \begin{vmatrix} te^{2t} & e^{2t} \\ e^{2t} + 2te^{2t} & 2e^{2t} \end{vmatrix} = -e^{4t} \neq 0$$
Hence these functions are LI.

Section 4.3
In problems 2, 4, 6 and 8, the auxilliary equation for the given differential equation has complex roots. Find a general solution.

2.) $y'' + 9y = 0$
Auxiliary equation
$r^2 + 9 = 0$
$r = \pm 3i$
$\alpha = 0, \beta = 3$
$y(t) = c_1e^{(0)t}\cos 3t + c_2e^{(0)t}\sin 3t$

$y(t) = c_1 \cos 3t + c_2 \sin 3t$
4.) $z'' - 6z' + 10z = 0$
Auxiliary equation
$r^2 - 6r + 10 = 0$
$r = 3 \pm i$
$\alpha = 3, \beta = 1$
$z = c_1 e^{3t} \cos t + c_2 e^{3t} \sin t$

6.) $w'' + 4w' + 6w = 0$
Auxiliary equation
$r^2 + 4r + 6 = 0$
Using the quadratic equation $r = \frac{-b \pm \sqrt{b^2+4ac}}{2a}$
$r = \frac{-4 \pm \sqrt{16-24}}{2} = -2 \pm \sqrt{2}\, i$
$\alpha = -2, \beta = \sqrt{2}$
$w(t) = c_1 e^{-2t} \cos \sqrt{2}\, t + c_2 e^{-2t} \sin \sqrt{2}\, t$
8.) $4y'' - 4y' + 26y = 0$
Auxiliary equation
$2r^2 - 2r + 13 = 0$
$r = \frac{1}{2} \pm \frac{5}{2} i$
$\alpha = \frac{1}{2}, \beta = \frac{5}{2}$
$y(t) = c_1 e^{t/2} \cos 5t/2 + c_2 e^{t/2} \sin 5t/2$

In problem 17, find a general solution.
17.) $y'' - y' + 7y = 0$
Auxiliary equation
$r^2 - r + 7 = 0$, Solution is: $\frac{3}{2} i\sqrt{3} + \frac{1}{2}, \frac{1}{2} - \frac{3}{2} i\sqrt{3}$
$r = \frac{1}{2} \pm \frac{3\sqrt{3}}{2} i$
$\alpha = 1/2, \beta = 3\sqrt{3}/2$
$y(t) = c_1 e^{t/2} \cos(\frac{3\sqrt{3}}{2} t) + c_2 e^{t/2} \sin(\frac{3\sqrt{3}}{2} t)$

In problem 27, solve the given initial value problem.
27.) $y''' - 4y'' + 7y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$
Auxiliary equation.
$r^3 - 4r^2 + 7r - 6 = 0$
Examining the divisors of –6, that is, $\pm 1, \pm 2, \pm 3, \pm 6,$ we find that $r = 2$ satisfies the equation. Next, we divide $r^3 - 4r^2 + 7r - 6 = 0$ by $r - 2$
$r^3 - 4r^2 + 7r - 6 = (r-2)(r^2 - 2r + 3) \Rightarrow r = 2; r\frac{2 \pm \sqrt{4-12}}{2} = 1 \pm \sqrt{2}\, i$
A general solution to the differential equation is given by
$y(t) = c_1 e^{2t} + c_2 e^t \cos \sqrt{2}\, t + c_3 e^t \sin \sqrt{2}\, t$

$\Rightarrow$

$$y' = 2c_1e^{2t} + c_2e^t\cos\sqrt{2}\,t - \sqrt{2}\,c_2e^t\sin\sqrt{2}\,t + c_3e^t\sin\sqrt{2}\,t + \sqrt{2}\,c_3e^t\cos\sqrt{2}\,t$$
$$= 2e^{2t}c_1 + e^tc_2(\cos t\sqrt{2} - \sqrt{2}\,\sin t\sqrt{2}) + e^tc_3(\sin t\sqrt{2} + \sqrt{2}\,\cos t\sqrt{2});$$
$$y'' = 4c_1e^{2t} + c_2e^t(-\cos\sqrt{2}\,t - 2\sqrt{2}\,\sin\sqrt{2}\,t) + c_3e^t(-\sin\sqrt{2}\,t + 2\sqrt{2}\,\cos\sqrt{2}\,t)$$

$\Rightarrow$

$$c_1 + c_2 = 1;$$
$$2c_1 + c_2 + \sqrt{2}\,c_3 = 0;$$

$4c_1 - c_2 + 2\sqrt{2}\,c_3 = 0 \qquad \Rightarrow c_1 = 1; \qquad c_2 = 0; \qquad c_3 = -\sqrt{2}$

$\Rightarrow$

$$y(t) = e^{2t} - \sqrt{2}\,e^t\sin\sqrt{2}\,t$$

In problem 29b, find a general solution to the higher-order equation.

29.) b.

$y''' + 2y'' + 5y' - 26y = 0$

Auxiliary equation:

$r^3 + 2r^2 + 5r - 26 = 0$

Examining the divisors of -26, that is, $\pm1, \pm2, \pm13$, we find that $r = 2$ satisfies the equation. Next, we divide $y^3 + 2y^2 + 5y - 26$ by $r - 2$

$r^3 + 2r^2 + 5r - 26 = (r-2)(r^2 + 4r + 13) \qquad \Rightarrow r = 2; \qquad r = -2 \pm 3i \qquad \Rightarrow$

A general solution to the differential equation is given by

$y(t) = c_1e^{2t} + c_2e^{-2t}\cos 3t + c_3e^{-2t}\sin 3t$