MA 221 Homework Solutions
Due date: February 24, 2015

4.4 pg. 182 # 6, 14, 12, 13, 15, 17, 21, 23
(Underlined Problems are to be handed in)

In problem 5 determine whether the method or not the method of undetermined coefficients can be applied to find a particular solution.

6) $y''(\theta) + 3y'(\theta) - y(\theta) = \sec\theta$

Since $\sec\theta = \frac{1}{\cos\theta}$, the method of undetermined coefficients is not applicable.
In problems 11, 12, 13, 15, 17, 21 and 23 find a particular solution to the differential equation.

12.)

$$2x' + x = 3t^2$$

$2x' + x = 0$, Exact solution is: $\left\{C_3 e^{-\frac{1}{2}t}\right\}$. Thus t^2 is not a homogeneous solution.
$x_p = (At^2 + Bt + C)$
$x_p' = 2At + B$
$2(2At + B) + At^2 + Bt + C = 3t^2$
$At^2 + (4A + B)t + (2B + C) = 3t^2$
$A = 3; 4A + B = 0; 2B + C = 0$
$A = 3; B = -12, C = 24$
$x_p = 3t^2 - 12t + 24$

13.) $y'' - y' + 9y = 3\sin 3t$

$$p(r) = r^2 - r + 9$$

$p(r) = 0 \Rightarrow r^2 - r + 9 = 0$, Solution is: $\frac{1}{2}i\sqrt{35} + \frac{1}{2}, \frac{1}{2} - \frac{1}{2}i\sqrt{35}$

Thus $\sin 3t$ is not a homogeneous solution

Method I: We assume

$$y_p = A\sin 3t + B\cos 3t$$
$$y_p' = 3A\cos 3t - 3B\sin 3t$$
$$y_p'' = -9A\sin 3t - 9B\cos 3t$$

Plugging into the DE we have
$-9A\sin 3t - 9B\cos 3t - 3A\cos 3t + 3B\sin 3t + 9A\sin 3t + 9B\cos 3t = 3\sin 3t$
$3B\sin 3t - 3A\cos 3t = 3\sin 3t$

$A = 0, B = 1$
$y_p = \cos 3t$

Method II: Consider a companion equation $v'' - v' + 9v = 3\cos 3t$.
Multiply the original DE by i and add it to the companion equation, and let $w = iy + v$. Then we have

$$w'' - w' + 9w = 3(i\sin 3t + \cos 3t) = 3e^{3it}$$

$p(\lambda) = \lambda^2 - \lambda + 9$ so $p(3i) = -9 - 3i + 9 = 3i \neq 0$.
Therefore

$$w_p = \frac{3e^{3it}}{3i} = -i(\cos 3t + i\sin 3t)$$

Now

$$y_p = \operatorname{Im} w_p = \cos 3t$$

15.)

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = xe^x$$

Here $p(r) = r^2 - 5r + 6 = (r-3)(r-2)$. Thus $r = 2, 3$ and the homogeneous solution is

$$y_h = c_1e^{2x} + c_2e^{3x}$$

Since e^x is not a homogeneous solution, we assume

$$y_p = (Ax + B)e^x$$

Taking derivatives and substituting into the DE leads to
$y_p' = (Ax + B + A)e^x$
$y_p'' = (Ax + B + 2A)e^x$
$(Ax + B + 2A)e^x - 5(Ax + B + A)e^x + 6(Ax + B)e^x = xe^x$
$(2Ax - 3A + 2B)e^x = xe^x$
$2A = 1 \Rightarrow A = 1/2$
$-3A + 2B = 0 \Rightarrow B = 3/4$

$$y_p = (\frac{x}{2} + \frac{3}{4})e^x$$

16) $\theta''(t) - \theta(t) = t\sin t$
$r^2 - 1 = 0 \quad\quad \Rightarrow \quad\quad r = \pm 1$
$\theta_p(t) = (A_1t + A_0)\cos t + (B_1t + B_0)\sin t$
$\theta_p'(t) = A_1\cos t - (A_1t + A_0)\sin t + B\sin t + (B_1t + B_0)\cot s$
$\quad\quad = (B_1t + A_1 + B_0)\cos t + (-A_1t - A_0 + B_1)\sin t$
$\theta_p''(t) = B_1\cos t - (B_1t + B_0 + A_1)\sin t - A_1\sin t + (-A_1t - A_0 + B_1)\cos t$
$\quad\quad = (-A_1t - A_0 + B_1)\cos t + (-B_1t - B_0 - 2A_1)\sin t$
Substituting into original equation: $\theta''(t) - \theta(t)$
$\theta_p'' - \theta_p = (-A_1t - A_0 + B_1)\cos t + (-B_1t - B_0 - 2A_1)\sin t + (A_1t + A_0)\cos t + (B_1t + B_0)\sin t$
$\Rightarrow -2A_1t\cos t + (-2A_0 + 2B_1)\cos t - 2B_1t\sin t + (-2A_1 - 2B_0)\sin t = t\sin t$
Equating Coefficients:

$-2A_1 = 0 \quad\Rightarrow\quad A_1 = 0$
$-2A_0 + 2B_1 = 0 \quad\Rightarrow\quad B_1 = A_0$
$-2B_1 = 1 \quad\Rightarrow\quad B_1 = -\frac{1}{2}$ and so, $A_0 = -\frac{1}{2}$
$-2A_1 - 2B_0 = 0 \quad\Rightarrow\quad B_0 = 0$

Therefore, the particular solution to the nonhomogeneous equation $\theta'' - \theta = t\sin t$ is given by:

$$\theta_p(t) = -\frac{t\sin t + \cos t}{2}$$

17)

$$y'' - 2y' + y = 8e^t$$

$p(r) = r^2 - 2r + 1 = (r-1)^2.$ Thus $r = 1$ is a repeated root and

$$y_h = c_1e^t + c_2te^t$$

$\alpha = 1$ and $p'(r) = 2r - 2 = 2(r-1).$ Thus $p(1) = p'(1) = 0.$ Thus

$$y_p = \frac{Kt^2e^t}{p''(\alpha)} = \frac{8t^2e^t}{2} = 4t^2e^t$$

21.) $x''(t) - 4x'(t) + 4x(t) = te^{2t}$

Since

$$p(r) = r^2 - 4r + 4 = (r-2)^2$$

we see that $r = 2$ is a repeated root and

$$x_h = c_1e^{2t} + c_2te^{2t}$$

Hence we seek

$$x_p = t^2(At + B)e^{2t}$$

$x_p' = (3At^2 + 2Bt)e^{2t} + 2(At^3 + Bt^2)e^{2t}$
$x_p'' = (6At + 2B)e^{2t} + 4(3At^2 + 2Bt)e^{2t} + 4(At^3 + Bt^2)e^{2t}$
$(6At + 2B)e^{2t} + 4(3At^2 + 2Bt)e^{2t} + 4(At^3 + Bt^2)e^{2t} - 4[(3At^2 + 2Bt)e^{2t} + 2(At^3 + Bt^2)e^{2t}] + 4t$
$(6At + 2B)e^{2t} = te^{2t}$
$B = 0, A = 1/6$
$x_p = t^3e^{2t}/6$

22) $x''(t) - 2x'(t) + x(t) = 24t^2e^t$
$r^2 - 2r + 1 = 0$
$(r-1)(r-1) = 0$
$r = 1$ (double root $\Rightarrow s = 2$)
$x_p = t^2(At^2 + Bt + C)e^t$
$x_p = (At^4 + Bt^3 + Ct^2)e^t$
$x_p' = (4At^3 + 3Bt^2 + 2Ct)e^t + (At^4 + Bt^3 + Ct^2)e^t$
$x_p'' = (12At^2 + 6Bt + 2C)e^t + (8At^3 + 6Bt^2 + 4Ct)e^t + (At^4 + Bt^3 + Ct^2)e^t$
Substituting into original equation: $x''(t) - 2x'(t) + x(t)$
$(12At^2 + 6Bt + 2C)e^t + (8At^3 + 6Bt^2 + 4Ct)e^t + (At^4 + Bt^3 + Ct^2)e^t - (8At^3 + 6Bt^2 + 4Ct)e^t -$
$(12At^2 + 6Bt + 2C)e^t = 24t^2e^t$
$12At^2 + 6Bt + 2C = 24t^2$
$12A = 24 \quad\Rightarrow\quad A = 2$

$6B = 0 \quad \Rightarrow \quad B = 0$
$2C = 0 \quad \Rightarrow \quad C = 0$
$x_p = 2t^4e^t$

23)

$$y''(\theta) - 7y'(\theta) = \theta^2$$

Since there is no y in the equation, we seek

$$y_p = A_2\theta^3 + A_1\theta^2 + A_0\theta$$

Then

$$y_p' = 3A_2\theta^2 + 2A_1\theta + A_0$$

$$y_p'' = 6A_2\theta + 2A_1$$

Hence

$$y_p''(\theta) - 7y_p'(\theta) = 6A_2\theta + 2A_1 - 7(3A_2\theta^2 + 2A_1\theta + A_0) = \theta^2$$

Thus

$$-21A_2 = 1 \quad 6A_2 - 14A_1 = 0 \quad 2A_1 - 7A_0 = 0$$

This leads to

$$A_2 = -\frac{1}{21} \quad A_1 = \frac{3}{7}A_2 = -\frac{1}{49} \quad A_0 = \frac{2}{7}A_1 = -\frac{2}{343}$$

Finally

$$y_p = -\frac{1}{21}\theta^3 - \frac{1}{49}\theta^2 - \frac{2}{343}\theta$$