Ma 221 Homework Solutions Due Date: Feb. 3, 2015

2.3 p.51-53 #2, <u>4</u>, 6, 8, <u>9</u>, 11, <u>19</u>, <u>30</u>; §2.6 p.74 #21, <u>23</u>, <u>28</u> (Underlined problems are handed in)

2.3 p.51 to p. 53 #2, <u>4</u>, 6, 8, <u>9</u>, 10, 11, 13, 15, <u>17</u>, 18, <u>19</u>, 21, 22, <u>30</u> (Underlined Problems are handed in)

For problems 1, 3, and 5 determine whether the given equation is separable, linear, neither, or both.

2.) $x^2\frac{dy}{dx} + \sin x = y$

Isolating $\frac{dy}{dx}$ we get:
$\frac{dy}{dx} = \frac{y - \sin x}{x^2}$. Since the right hand side cannot be represented as a product $g(x)p(y)$, the equation is **not separable**.

4.) $(t^2+1)\frac{dy}{dt} = yt - y$

This is a linear equation with independent variable t and dependent variable y. This is a **separable** equation as shown:

Isolating $\frac{dy}{dt}$ we get:

$$\frac{dy}{dt} = \frac{y(t-1)}{(t^2+1)} \Rightarrow \qquad \frac{dy}{y} = \frac{(t-1)}{(t^2+1)}dt \Rightarrow \qquad \frac{dy}{dt} = \frac{(t-1)}{(t^2+1)}y = g(t)p(y)$$

6.) $x\frac{dx}{dt} + t^2 x = \sin t$

In this equation, the independent variable is t and the dependent variable is x. Dividing by x we obtain
$\frac{dx}{dt} = \frac{\sin t}{x} - t^2$.
Therefore, it is neither linear, because of the $\frac{\sin t}{x}$ term, nor separable, because the right-hand side is not a product of functions of single variables x and t.

For problems 8, 9, 11, 13 and 15, obtain the general solution to the equation

8.) $\frac{dy}{dx} - y - e^{3x} = 0$

In this equation, $P(x) = -1$ and $Q(x) = e^{3x}$
The integrating factor is thus: $\mu(x) = \exp(\int P(x)dx) = \exp(\int(-1)dx) = e^{-x}$
Multiplying both sides of the equation by $\mu(x)$ and integrating yields:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{3x} = e^{2x} \Rightarrow \frac{d(e^{-x}y)}{dx} = e^{2x}$$

$$\Rightarrow e^{-x}y = \int e^{2x}dx = \tfrac{1}{2}e^{2x} + C \Rightarrow \boxed{y = (\tfrac{1}{2}e^{2x} + C)e^{x} = \frac{e^{3x}}{2} + Ce^{x}}$$

9.) $x\frac{dy}{dx} + 2y = x^{-3}$

Putting the equation in standard form:

$\frac{dy}{dx} + \frac{2}{x}y = x^{-4}$

Find $\mu(x) = e^{\int P dx} = e^{\int (\frac{2}{x})dx} = e^{(2\ln|x|)} = (|x|)^2 = x^2$

Multiply through by $\mu(x)$ to get

$x^2\frac{dy}{dx} + 2xy = x^{-2} \Rightarrow \qquad \frac{d}{dx}yx^2 = x^{-2}$

Integrate to get

$\int \frac{d}{dx}yx^2 = \int x^{-2}$

$yx^2 = -x^{-1} + C$

Solve explicitly for y

$\Rightarrow \boxed{y = -x^{-3} + Cx^{-2}}$

11.) $(t + y + 1)dt - dy = 0$

Choosing t as the independent variable and y as the dependent variable, the equation can be put into standard form:

$$t + y + 1 - \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} - y = t + 1$$

Thus:$P(t) = -1$ and $\mu(t) = \exp\left[\int(-1)dt\right] = e^{-t}$

Multiplying both sides by $\mu(t)$ and integrating yields:

$e^{-t}\frac{dy}{dt} - e^{-t}y = (t+1)e^{-t} \Rightarrow \frac{d(e^{-t}y)}{dt} = (t+1)e^{-t}$

$\Rightarrow e^{-t}y = \int(t+1)e^{-t}dt = -(t+1)e^{-t} + \int e^{-t}dt = -(t+1)e^{-t} - e^{-t} + C = -(t+2)e^{-t} + C$

$\Rightarrow \boxed{y = e^{t}(-(t+2)e^{-t} + C) = -t - 2 + Ce^{t}}$

13.) $y\frac{dx}{dy} + 2x = 5y^3$

In this problem, the independent variable is y and the dependent variable is x. So, we divide the equation by y to rewrite it in standard form.

$$y\frac{dx}{dy} + 2x = 5y^3 \Rightarrow \qquad \frac{dx}{dy} + \frac{2}{y}x = 5y^2$$

Therefore, $P(y) = \frac{2}{y}$ and the integrating factor, $\mu(y)$, is

$$\mu(y) = e^{\int \frac{2}{y}dy} = e^{2\ln|y|} = |y|^2 = y^2$$

Multiplying the equation (in standard form) by y^2 and integrating yield

$\left(\frac{dx}{dy} + \frac{2}{y}x = 5y^2\right)y^2 = y^2\frac{dx}{dy} + 2yx = 5y^4 \Rightarrow \qquad \frac{d}{dy}(y^2x) = 5y^4$

$\Rightarrow \qquad y^2x = \int 5y^4 dy = y^5 + C$

$\Rightarrow \boxed{x = y^{-2}(y^5 + C) = y^3 + Cy^{-2}}$

15.) $(x^2+1)\frac{dy}{dx} + xy - x = 0$

Divide by (x^2+1)

$\frac{dy}{dx} + \frac{x}{x^2+1}y = \frac{x}{x^2+1}$

so $P(x) = \frac{x}{x^2+1}$

Find $\mu(x) = e^{\int P(x)dx}$

$\mu(x) = e^{\int \frac{x}{x^2+1}} = e^{\frac{1}{2}\ln(x^2+1)} = (x^2+1)^{(1/2)}$

Multiply through by $\mu(x)$ to get

$(x^2+1)^{\frac{1}{2}}y' + \frac{x}{(x^2+1)^{\frac{1}{2}}}y = \frac{x}{(x^2+1)^{\frac{1}{2}}}$ or

$\frac{d}{dx}\left((x^2+1)^{\frac{1}{2}}y\right) = \frac{x}{(x^2+1)^{\frac{1}{2}}}$

Integrating gives

$y(x^2+1)^{(1/2)} = (x^2+1)^{(1/2)} + C$

Solve explicitly for y

$y = 1 + C(x^2+1)^{-(1/2)}$

For problems 17, 18 and 19, 21, 22 solve the initial value problems.

17.)

$$\frac{dy}{dx} - \frac{y}{x} = xe^x \qquad y(1) = e - 1$$

This is a linear equation with $P(x) = -1/x$ and $Q(x) = xe^x$. The integrating factor is given by:

$\mu(x) = e^{\int Pdx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$; $\qquad$ For $x > 0$

Multiply through by $\mu(x)$ to get

$\frac{1}{x}y' - \frac{y}{x^2} = e^x$

or

$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = e^x$

Integrate to get

$\frac{y}{x} = e^x + C$

Solve explicitly for y

$y = xe^x + Cx$

Plug in initial condition $y(1) = e - 1$ and solve for C

$e - 1 = e + C \Rightarrow \qquad C = -1$

Plug in the value for C

$y = xe^x - x$

Using SNB to check our answer we have

$\begin{array}{c} \frac{dy}{dx} - \frac{y}{x} = xe^x \\ y(1) = e - 1 \end{array}$, Exact solution is: $xe^x - x$

18.) $\frac{dy}{dx} + 4y - e^{-x} = 0$; $y(0) = \frac{4}{3}$

$\frac{dy}{dx} + 4y = e^{-x}$

Find $\mu(x) = e^{\int Pdx} = e^{\int 4dx} = e^{4x}$

Multiply through by $\mu(x)$ to get

$e^{4x}y' + 4e^{4x}y = e^{3x}$

or

$\frac{d}{dx}(e^{4x}y) = e^{3x}$

Integrate to get

$ye^{4x} = \frac{1}{3}e^{3x} + c$

Solve explicitly for y

$y = \frac{1}{3}e^{-x} + \frac{C}{e^{4x}}$

Plug in initial condition $y(0) = \frac{4}{3}$ and solve for C

$\frac{4}{3} = \frac{1}{3}e^{-0} + \frac{C}{e^{4(0)}}$

$\frac{4}{3} = \frac{1}{3} + C$

So $C = 1$

Plug in the value for C

$y = \frac{1}{3}e^{-x} + e^{-4x}$

Using SNB to check our answer we have

$\begin{array}{c} \frac{dy}{dx} + 4y - e^{-x} = 0 \\ y(0) = \frac{4}{3} \end{array}$, Exact solution is: $y(x) = \frac{1}{3}e^{-x} + e^{-4x}$

19.)

$$t^2\frac{dx}{dt} + 3tx = t^4 \ln t + 1, \qquad x(1) = 0$$

In this problem, t is the dependent variable and x is the dependent variable. Dividing by t^2 we have

$$\frac{dx}{dt} + \frac{3}{t}x = t^2 \ln t + \frac{1}{t^2}$$

The integrating factor is $e^{\int \frac{3}{t}dt} = t^3$. Thus

$$t^3\frac{dx}{dt} + 3t^2x = t^5\ln t + t$$

or

$$\frac{d}{dt}(t^3x) = t^5\ln t + t$$

Integrating we have since $\int t^5\ln t dt = \frac{1}{6}t^6\ln t - \frac{1}{36}t^6 + c$

$$t^3x = \frac{1}{6}t^6\ln t - \frac{1}{36}t^6 + \frac{t^2}{2} + c$$

The IC implies

$$0 = -\frac{1}{36} + \frac{1}{2} + c$$

so $c = -\frac{17}{36}$ and

$$t^3x = \frac{1}{6}t^6\ln t - \frac{1}{36}t^6 + \frac{t^2}{2} - \frac{17}{36}$$

or

$$x = \frac{1}{6}t^3\ln t - \frac{1}{36}t^3 + \frac{1}{2t} - \frac{17}{36t^3}$$

21.) $\cos x\frac{dy}{dx} + y\sin x = 2x\cos^2 x, \qquad y(\frac{\pi}{4}) = \frac{-15\sqrt{2}\pi^2}{32}$

Putting the equation in standard form:

$\frac{dy}{dx} + \frac{\sin x}{\cos x}y = 2x\cos x \Rightarrow \qquad \frac{dy}{dx} + (\tan x)y = 2x\cos x$

Find $\mu(x) = e^{\int Pdx} = e^{\int \tan x dx} = e^{(-\ln|\cos x|)} = |\cos x|^{-1}$

At the initial point, $x = \frac{\pi}{4}, \cos\frac{\pi}{4} > 0$ therefore we can take $\mu(x) = (\cos x)^{-1}$

Multiply through by $\mu(x)$ to get

$\frac{1}{\cos x}\frac{dy}{dx} + \frac{\sin x}{\cos^2 x}y = 2x \qquad \Rightarrow \qquad \frac{d}{dx}(\frac{y}{\cos x}) = 2x$

Integrate to get

$\int \frac{d}{dx}(\frac{y}{\cos x}) = \int 2x$

$\frac{y}{\cos x} = x^2 + C$

Solve explicitly for y

$y = x^2\cos x + C\cos x$

Plug in initial condition $y(\frac{\pi}{4}) = \frac{-15\sqrt{2}\pi^2}{32}$ and solve for C

$\frac{-15\sqrt{2}\pi^2}{32} = \frac{\pi}{4}^2\cos\frac{\pi}{4} + C\cos\frac{\pi}{4}$

So $C = -\pi^2$

Plug in the value for C

$y = x^2\cos x - \pi^2\cos x = \cos x(x^2 - \pi^2)$

22.) $\sin x\frac{dy}{dx} + y\cos x = x\sin x, \qquad y(\frac{\pi}{2}) = 2$

Putting the equation in standard form:

$\frac{dy}{dx} + \frac{\cos x}{\sin x}y = x \Rightarrow \qquad \frac{dy}{dx} + (\cot x)y = x$

Find $\mu(x) = e^{\int Pdx} = e^{\int\left(\frac{\cos x}{\sin x}\right)dx} = e^{(\ln(\sin x))} = \sin x$

Multiply through by $\mu(x)$ to get

$\sin x\frac{dy}{dx} + y\cos x = x\sin x \Rightarrow \qquad \frac{d}{dx}(y\sin x) = x\sin x$

Integrate to get

$\int \frac{d}{dx}(y\sin x) = \int x\sin x$ (Using integration by parts)

$y\sin x = \sin x - x\cos x + C$

Solve explicitly for y

$y = 1 - x\frac{\cos x}{\sin x} + \frac{C}{\sin x} = 1 - x\cot x + \frac{C}{\sin x}$

Plug in initial condition $y(\frac{\pi}{2}) = 2$ and solve for C

$2 = 1 - \frac{\pi}{2}\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} + \frac{C}{\sin\frac{\pi}{2}}$

So $C = 1$

Plug in the value for C

$y = 1 - x\cot x + \frac{1}{\sin x} = 1 - x\cot x + \csc x$

30.) Show that the substitution $v = y^3$ reduces equation $\frac{dy}{dx} + 2y = xy^{-2}$ to the equation $\frac{dv}{dx} + 6v = 3x$. Then solve the equation for v, and make the substitution $v = y^3$ to obtain the solution the equation $\frac{dy}{dx} + 2y = xy^{-2}$.

$\frac{dy}{dx} + 2y = xy^{-2}$

Divide by y^{-2}

$y^2\frac{dy}{dx} + 2y^3 = x$

$v = y^3$

Differentiate v with respect to x

$\frac{dv}{dx} = 3y^2\frac{dy}{dx}$

Divide by 3

$\frac{1}{3}\frac{dv}{dx} = y^2\frac{dy}{dx}$

Notice that $\frac{1}{3}\frac{dv}{dx}$ is equal to the first term on the left hand side of the equation. Make that substitution.

$\frac{1}{3}\frac{dv}{dx} + 2v = x$

Now multiply by 3 to get a first order linear differential equation.

$\frac{dv}{dx} + 6v = 3x$

........(First order linear differential equation)

Find $\mu(x)$

$\mu(x) = e^{\int 6dx} = e^{6x}$

Multiply through by $\mu(x)$ to get

$e^{6x}v' + 6e^{6x}v = \frac{d}{dx}(e^{6x}v) = 3xe^{6x}$Now integrate
$ve^{6x} = \int e^{6x}(3x)dx = \frac{1}{2}xe^{6x} - \frac{1}{12}e^{6x} + C$
Where SNB was used to evaluate the integral.
Solve explicitly for v yields
$v = \frac{1}{2}x - \frac{1}{12} + Ce^{-6x}$
Plug y back into the equation from the substitution $v = y^3$ and solve for y.
$y^3 = \frac{1}{2}x - \frac{1}{12} + Ce^{-6x}$
or
$y = (\frac{1}{2}x - \frac{1}{12} + Ce^{-6x})^{\frac{1}{3}}$

2.6 p.74 # 21, 23, 28
For 21, 23 and 28 use the method discussed under "Bernoulli Equations" to solve the problems.

21.)
$\frac{dy}{dx} + \frac{y}{x} = x^2y^2$
This is a Bernoulli Equation with $n = 2$
Let
$u = y^{1-n} = y^{1-2} = y^{-1}$
then $\quad y = u^{-1} \quad \Rightarrow y' = -u^2u'$
and the last equation becomes
$-\frac{1}{u^2}\frac{du}{dx} + \frac{1}{ux} = \frac{x^2}{u^2} \quad \Rightarrow \quad \frac{du}{dx} - \frac{1}{x}u = -x^2$
This is a linear equation with $P(x) = -\frac{1}{x}$
Find $\mu(x) = e^{\int Pdx} = e^{\int(-\frac{1}{x})dx} = e^{(-\ln|x|)} = x^{-1}$
Multiply through by $\mu(x)$ to get
$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -x \Rightarrow \quad \frac{d}{dx}(\frac{1}{x}u) = -x$
Integrate to get
$\int \frac{d}{dx}(\frac{1}{x}u) = \int -xdx$
$\frac{1}{x}u = \frac{-x^2}{2} + C_1 \Rightarrow \quad u = -\frac{1}{2}x^3 + C_1x$
Solve explicitly for y
$y = \frac{1}{\frac{x^3}{2} + C_1x} = \frac{2}{Cx - x^3}$
$y = 0$ is also a solution to the original equation. It was lost in the first step when we multiplied by u^2 (also same as dividing by y^2).
23.)
$\frac{dy}{dx} = \frac{2y}{x} - x^2y^2$
or after dividing by y^2 and moving the first term on the right to the left we have
$y^{-2}\frac{dy}{dx} - \frac{2y^{-1}}{x} = -x^2$

Let
$v = y^{-1}$
$v' = -y^{-2}y'$
and the last equation becomes
$v' + 2\frac{v}{x} = x^2$
This is a first order linear equation in v.
The integrating faction for this equation is
$e^{\int \frac{2}{x}dx} = x^2$
Multiplying the DE by this gives
$x^2v' + 2xv = \frac{d}{dx}(x^2v) = x^4$
Thus
$x^2v = \frac{1}{5}x^5 + c_1$
and
$\frac{1}{y} = v = \frac{x^3}{5} + \frac{c_1}{x^2}$
Therefore
$y = \left(\frac{x^5+5c_1}{5x^2}\right)^{-1}$
Letting $C = 5c_1$ we have finally
$y = \left(\frac{5x^2}{x^5+C}\right)$
$y = 0$ is also a solution to the original equation. It was lost in the first step when we divided by y^2.

28.)
$\frac{dy}{dx} + y^3x + y = 0$
Rewrite the equation as
$y' + y = -xy^3$
Multiply both sides by y^{-3} to get
$y^{-3}y' + y^{-2} = -x$
Let
$v = y^{-2}$
$v' = -2y^{-3}y'$
The DE then can be written as
$-\frac{v'}{2} + v = -x$
or
$v' - 2v = 2x$
This is a first order linear equation in v. The integrating factor is
$e^{\int -2dx} = e^{-2x}$
Multiplying the last equation by this leads to
$e^{-2x}v' - 2e^{-2x}v = \frac{d}{dx}(e^{-2x}v) = 2xe^{-2x}$

Integrating gives
$e^{-2x}v = \int(2xe^{-2x})dx = -\frac{1}{2}e^{-2x} - xe^{-2x} + c$
Thus
$\frac{1}{y^2} = -\frac{1}{2} - x + ce^{2x}$
Using SNB to check, we have
$y' + y = -xy^3$, Exact solution is: $\frac{1}{\sqrt{C_1 e^{2x} - x - \frac{1}{2}}}$