MA 221 Homework Solutions
Due date: February 5, 2015

pg. 61 - 62 Sec. 2.4 #10, <u>11</u>, <u>13</u>, 15, <u>17</u>, 19, <u>23</u>, 24, <u>25</u>, <u>27a</u>, <u>29</u>
(Underlined Problems are to be turned in.)

In problems 9, 11, 13, 15, 17 and 19, determine whether the equation is exact. If it is, then solve it.

10.) $(2xy+3)dx+(x^2-1)dy = 0$
Computing partial derivatives we obtain:
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy+3) = 2x$
$\frac{\partial N}{dx} = \frac{\partial}{\partial x}(x^2-1) = 2x \qquad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.
Integration yields:
$F(x,y) = \int M(x,y)dx = \int(2xy+3)dx = x^2y+3x+f(y)$
$\frac{\partial F}{\partial y} = x^2+f'(y) = x^2-1 \qquad \Rightarrow f'(y) = -1 \qquad \Rightarrow f(y) = \int -1dy = -y+C \qquad \Rightarrow$general solution is
$F(x,y) = x^2y+3x-y = C$ or explicitly, $y = \frac{C-3x}{x^2-1}$

11.) $(\cos x\cos y+2x)dx-(\sin x\sin y+2y)dy = 0$
Computing partial derivatives we obtain:
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\cos x\cos y+2x) = -\cos x\sin y,$
$\frac{\partial N}{dx} = \frac{\partial}{\partial x}(-(\sin x\sin y+2y)) = -\cos x\sin y \qquad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.
Integration yields:
$F(x,y) = \int M(x,y)dx = \int(\cos x\cos y+2x)dx = \sin x\cos y+x^2+f(y)$
$\frac{\partial F}{\partial y} = -\sin x\sin y+f'(y) = -\sin x\sin y-2y \qquad \Rightarrow f'(y) = -2y$
$\Rightarrow f(y) = \int -2ydy = -y^2+c \qquad \Rightarrow$general solution is

$$F(x,y) = \sin x\cos y+x^2-y^2 = C$$

13.) $(\frac{t}{y})dy+(1+\ln y)dt = 0$
Computing partial derivatives we obtain:
$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}(\frac{t}{y}) = \frac{1}{y}$
$\frac{\partial N}{dy} = \frac{\partial}{\partial y}(1+\ln y) = \frac{1}{y} \qquad \Rightarrow \frac{\partial M}{\partial t} = \frac{\partial N}{\partial y}$ the equation is exact.
Integration yields:
$F(y,t) = \int M(y,t) = \int(\frac{t}{y})dy = t\ln y+f(t) = t\ln y+f(t)$
$\frac{\partial F}{\partial t} = \ln y+f'(t) = 1+\ln y \qquad \Rightarrow f'(t) = 1$
$\Rightarrow f(t) = \int 1dt = t+c$
general solution is:
$F(y,t) = t\ln y+t = C$, or explicitly, $t = \frac{C}{1+\ln y}$

15.) $\cos\theta dr - (r\sin\theta - e^{\theta})d\theta = 0$

Computing partial derivatives we obtain:

$\frac{\partial M}{\partial \theta} = \frac{\partial}{\partial \theta}(\cos\theta) = -\sin\theta$

$\frac{\partial N}{dr} = \frac{\partial}{\partial r}(-(r\sin\theta - e^{\theta})) = \sin\theta \qquad \Rightarrow \frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r}$ the equation is exact.

Integration yields:

$F(\theta, r) = \int M(\theta, r) = \int(\cos\theta)dr = r\cos\theta + f(\theta)$

$\frac{\partial F}{\partial \theta} = -r\sin\theta + f'(\theta) = -r\sin\theta + e^{\theta} \qquad \Rightarrow f'(\theta) = e^{\theta} \qquad \Rightarrow f(\theta) = \int e^{\theta}d\theta = e^{\theta} + C$

general solution is:

$F(\theta, r) = r\cos\theta + e^{\theta} = C$, or explicitly, $\qquad r = \frac{C-e^{\theta}}{\cos\theta} = (C - e^{\theta})\sec\theta$

17.) $(\frac{1}{y})dx - \left(3y - \frac{x}{y^2}\right)dy = 0$

$M_y = -\frac{1}{y^2} \qquad N_x = \frac{1}{y^2} \qquad M_y \neq N_x$ the equation is not exact.

19.) $\left(2x + \frac{y}{1+x^2y^2}\right)dx + \left(\frac{x}{1+x^2y^2} - 2y\right)dy = 0$

Computing partial derivatives we obtain:

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2x + \frac{y}{1+x^2y^2}\right) = \frac{1+x^2y^2-2x^2y^2}{(1+x^2y^2)^2}$

$\frac{\partial N}{dx} = \frac{\partial}{\partial x}\left(\frac{x}{1+x^2y^2} - 2y\right) = \frac{1+x^2y^2-2x^2y^2}{(1+x^2y^2)^2} \qquad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

Integration yields:

$F(x,y) = \int M(x,y)dx = \int\left(2x + \frac{y}{1+x^2y^2}\right)dx = x^2 + \arctan xy + f(y)$

$\frac{\partial F}{\partial y} = \frac{x}{1+(xy)^2} + f'(y) = \left(\frac{x}{1+x^2y^2} - 2y\right) \qquad \Rightarrow f'(y) = -2y \qquad \Rightarrow f(y) = \int -2ydy = -y^2 + C$

solution is

$F(x,y) = x^2 + \arctan xy - y^2 = C$

In problems 23, 24, and 25, solve the initial value problem.

23.) $(e^ty + te^ty)dt + (te^t + 2)dy = 0 \qquad y(0) = -1$

$M_y = e^t + te^t \qquad N_t = te^t + e^t \qquad \Rightarrow M_y = N_t$ the equation is exact.

$F(y,t) = \int M_y dt = \int N_t dy = \int(te^t + 2)dy = tye^t + 2y + f(t)$

$\frac{\partial F}{\partial t} = ye^t + tye^t + f'(t) = e^ty + te^ty \qquad \Rightarrow f'(t) = 0 \qquad \Rightarrow F(y,t) = tye^t + 2y + c$

$\Rightarrow tye^t + 2y = C \qquad \Rightarrow y = C/(te^t + 2)$

Using the initial condition:

$y(0) = C/(0+2) = -1 \qquad \Rightarrow C = -2 \qquad \Rightarrow y = -2/(te^t + 2)$

24)

$$(e^tx + 1)dt + (e^t - 1)dx = 0 \quad x(1) = 1$$

$$M_x = e^t = N_t$$

Therefore the equation is exact. Thus there exists $f(t,x)$ such that

$$f_t = e^tx + 1 \text{ and } f_x = e^t - 1$$

Integrating the second equation w.r.t. to x we have

$$f(t,x) = xe^t - x + g(t)$$

Differentiating this w.r.t. t yields$f_t = xe^t + g'(t) = M = e^t x + 1$Thus $g(t) = t + C$ and

$$f(t,x) = xe^t - x + t + C$$

so the solution is given by

$$xe^t - x + t = K$$

The initial condition implies

$$e - 1 + 1 = K$$

so the solution is given by

$$xe^t - x + t = e$$

25.) $(y^2 \sin x)dx + (\frac{1}{x} - \frac{y}{x})dy = 0, \quad y(\pi) = 1$

Notice this equation is not exact. It is separable.

$\int x \sin x dx = \int \frac{y-1}{y^2} dy$

$-x \cos x + \sin x = \ln y + \frac{1}{y} + C$

Substituting the initial condition

$-\pi \cos \pi + \sin \pi = \ln 1 + \frac{1}{1} + C$

$C = \pi - 1$

$-x \cos x + \sin x = \ln y + \frac{1}{y} + \pi - 1$

27.) Find the most general function $M(x,y)$ so that the equation is exact.

(a) $M(x,y)dx + (\sec^2 y - \frac{x}{y})dy = 0$

We want to find $M(x,y)$ so that for $N(x,y) = (\sec^2 y - \frac{x}{y})$ we have

$M_y(x,y) = N_x(x,y) = -\frac{1}{y}$

Therefore, we must integrate this last expression with respect to y:

$M(x,y) = \int(-\frac{1}{y})dy = -\ln|y| + f(x),$

where $f(x)$ is the constant of integration, a function of only of x.

29.) Consider the equation

$$(y^2 + 2xy)dx - x^2 dy = 0$$

(a) Show that the equation is not exact.

$M_y = 2y + 2x, N_x = -2x$ =>not exact

(b) Show that multiplying both sides of the equation by y^{-2} yields a new equation that is exact

$y^{-2}(y^2 + 2xy)dx - x^2 y^{-2} dy = 0$

$(1 + \frac{2x}{y})dx - \frac{x^2}{y^2} dy = 0$

$M_y = \frac{-2x}{y^2}, N_x = \frac{-2x}{y^2}$ => *exact*

(c) Use the solution of the resulting exact equation to solve the original equation.

$F(x,y) = \int(1 + \frac{2x}{y})dx$

$= x + \frac{x^2}{y} + g(y)$
$F_y = -\frac{x^2}{y^2} + g'(y) = -\frac{x^2}{y^2}$
$g'(y) = 0$
$g(y) = 0$
$x + \frac{x^2}{y} = C$
$y = \frac{x^2}{C-x}$
(d) Where any solutions lost in the process?
Yes, by dividing both sides by y^2 we lost the solution $y = 0$.