
This text is based on the following books:

- "Introduction to Real Analysis" by A.N. Kolmogorov and S.V. Fomin
- "Linear Algebra and Analysis" by Marc Zamansky

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

66 Uniform continuity

The definition of continuity of a mapping $f$ of a metric space $(E, d)$ into a metric space $(F, \delta)$ is the same as that given in the case of two arbitrary topological spaces. However we can now define the concept of uniform continuity.

Definition 66.1. Let $f$ be a mapping of a metric space $(E, d)$ into a metric space $(F, \delta)$. $f$ is said to be uniformly continuous if for each $\varepsilon > 0$ we can find $\alpha(\varepsilon) > 0$ such that if $d(x, x') < \alpha$ then $\delta(f(x), f(x')) < \varepsilon$.

Example 66.1. If $f$ is uniformly continuous it is continuous, but the converse is false. For example, the continuous mapping of $\mathbb{R}^+$ into $\mathbb{R}^+$ defined by $x \to 1/x$.

There are cases where continuity implies uniform continuity. These depend on topological properties of the space $(E, d)$. For example, let us prove the following result:

Theorem 66.1. Let $f$ be a mapping of a compact metric space $(E, d)$ into a metric space $(F, \delta)$. if $f$ is continuous it is uniformly continuous.

Proof. Let $\varepsilon > 0$. To each $x \in E$ we assign an open ball $B(x, r_x)$ of center $x$ and radius $r_x$ such that if $x' \in B(x, r_x)$, $\delta(f(x'), f(x)) < \varepsilon/2$. This may be done since $f$ is continuous.

Consider the open balls $B(x, r_x/2)$. They cover $E$ and, since $E$ is compact, include a finite cover $B(x_i, r_{x_i}/2)$. 

1
Let
\[ m = \inf \left( r_{x_1} / 2 \right) \]
and consider two points \( x, x' \) of \( E \) such that \( d(x, x') < m \). The point \( x \) is contained in a certain ball \( B(x_i, r_{x_i}/2) \) and we have
\[ d(x', x_i) \leq d(x', x) + d(x, x') < m + r_{x_i}/2 \leq r_{x_i} \]
It follows that \( x' \in B(x_i, r_{x_i}) \). We now have
\[ \delta(f(x'), f(x)) < \delta(f(x'), f(x_i)) + \delta(f(x), f(x_i)) < \varepsilon \]
since \( x' \) and \( x \) belong to \( B(x_i, r_{x_i}) \).

**Example 66.2.** Let us consider a continuous function defined by distance. The distance \( d(x, A) \) of a point \( x \) from a subset \( A \) of a metric space \( E \) is defined by
\[ d(x, A) = \inf_{y \in A} d(x, y) \]
We will prove that the function \( x \to d(x, A) \) is uniformly continuous on \( E \) for every set \( A \).

\( d(x, A) \) being the lower bound of the \( d(x, u) \) for \( u \in A \), given \( \varepsilon > 0 \) there exists a \( u_0 \in A \) such that
\[ d(x, A) \leq d(x, u_0) < d(x, A) + \varepsilon \]
Let \( y \) be another point of \( E \). We have
\[ d(y, u_0) \leq d(y, x) + d(x, u_0) < d(y, x) + d(x, A) + \varepsilon \]
and since
\[ d(x, A) = \inf_{u \in A} d(y, u) \leq d(y, u_0) \]
it follows that
\[ d(y, A) \leq d(y, x) + d(x, A) + \varepsilon \]
Since \( \varepsilon \) is arbitrary there results
\[ d(y, A) \leq d(y, x) + d(x, A) \]
and interchanging \( x \) and \( y \)
\[ d(x, A) \leq d(x, y) + d(y, A) \]
Thus for each subset $A$ and arbitrary points $x, y$ of $E$ we have

$$|d(x, A) - d(y, A)| < d(x, y)$$

which establishes the uniform continuity of the function $x \to d(x, A)$.

67 Extension by continuity

The following question arises in a natural way. If $E$ and $F$ are two spaces, $A$ a dense subspace of $E$, and $\phi$ a continuous mapping of $A$ into $F$, is there a mapping $f$ of $E$ into $F$ which is continuous in $E$ and whose restriction to $A$ is $\phi$?

This question can be posed more graphically as follows: if $f$ is a continuous mapping of $E$ into $F$ and $A$ is dense in $E$ can $f$ be reconstituted from its restriction $\phi$ to $A$?

The solution of this problem is called the \textit{extension of $\phi$ from $A$ to $E$ by continuity}. This, in fact, can be done if we impose some quite general conditions satisfied by metric spaces (which are separated and normal).

We first prove the following statement:

\textbf{Proposition 67.1.} If $f$ and $g$ are two continuous mappings of a space $E$ into a separated space $F$, and are equal at the points of a dense subset $A$ of $E$, then they are equal everywhere in $E$.

\textit{Proof.} For, if $x \in E$ is adherent to $A$, $f(x)$, the limit of $f(\xi)$ when $\xi$ tends to $x$, is also the limit when $\xi$ tends to $x$ in $A$. Since $f(\xi) = g(\xi)$ for $\xi \in A$, and $F$ is separated, the limits of $f$ and $g$ at each point $x \in E$ are equal. \hfill $\square$

Now let $\phi$ be a mapping of a set $A$ which is dense in $E$, into a separated space $F$. In order to be able to extend $\phi$ to $E$ we must suppose that when $\xi \in A$ tends to $x \in E$, $\phi(\xi)$ has a limit, which we shall denote by $f(x)$. (If $E$ is a metric space we can say that for every sequence $(\xi_n)$ of elements of $A$ converging to $x \in E$, $\phi(\xi_n)$ must have a limit, and this limit must be the same for every such sequence $(\xi_n)$.) We now prove the following theorem:

\textbf{Theorem 67.1.} Let $A$ be a dense subspace of a space $E$, $F$ a normal space, and $\phi$ a mapping of $A$ into $F$ such that for every $x \in E$, $\phi(\xi)$ has a limit $f(x)$ in $F$ when $\xi \in A$ tends to $x$. Then the function $f$ is continuous in $E$.

\textit{Proof.} See the proof of Theorem 38.1 \hfill $\square$
68 Contraction mapping

68.1 The fixed point theorem

Let $A$ be a mapping of a metric space $R$ into itself. Then $x$ is called a fixed point of $A$ if $Ax = x$, i.e. $A$ carries $x$ into itself. Suppose there exists a number $\alpha < 1$ such that

$$\rho(Ax, Ay) \leq \alpha \rho(x, y)$$

for every pair of points $x, y \in R$. Then $A$ is said to be a contraction mapping. Every contraction mapping is automatically continuous, since it follows from the "contraction condition" ($\rho(Ax, Ay) \leq \alpha \rho(x, y)$) that $Ax_n \to Ax$ whenever $x_n \to x$.

**Theorem 68.1. (Fixed point theorem).** Every contraction mapping $A$ defined on a complete metric space $R$ has a unique fixed point.

**Proof.** Given an arbitrary point $x_0 \in R$, let

$$x_1 = Ax_0, \quad x_2 = Ax_1 = A^2x_0, \ldots, \quad x_n = Ax_{n-1} = A^nx_0, \ldots$$

where $A^2x = A(A(x))$, $A^3x = A(A^2x) = A(A(A(x)))$, etc.

Then the sequence $\{x_n\}$ is fundamental. In fact, assuming to be explicit that $n \leq n'$, we have

$$\rho(x_n, x_{n'}) = \rho(A^n x_0, A^{n'} x_0) \leq \alpha^n \rho(x_0, x_{n' - n})$$

$$\leq \alpha^n [\rho(x_0, x_1) + \rho(x_1, x_2) + \cdots + \rho(x_{n' - n - 1}, x_{n' - n})]$$

$$\leq \alpha^n \rho(x_0, x_1)[1 + \alpha + \alpha^2 + \cdots + \alpha^{n' - n - 1}]$$

$$< \alpha^n \rho(x_0, x_1) \frac{1}{1 - \alpha}$$

But the expression on the right can be made arbitrary small for sufficiently large $n$, since $\alpha < 1$. Since $R$ is complete, the sequence $\{x_n\}$, being fundamental, has a limit

$$x = \lim_{n \to \infty} x_n$$

Then, by continuity of $A$,

$$Ax = A \lim_{n \to \infty} x_n = \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} x_{n+1} = x$$

This proves the existence of a fixed point $x$. To prove the uniqueness of $x$ we note that is

$$Ax = x, \quad Ay = y$$

then by definition of contraction mapping

$$\rho(x, y) \leq \alpha \rho(x, y)$$

But then $\rho(x, y) = 0$ since $\alpha < 1$, and hence $x = y$\]
Remark. The fixed point theorem can be used to prove existence and uniqueness theorems for solutions of equations of various types. Besides showing that an equation of the form $Ax = x$ has a unique solution, the fixed point theorem also gives a practical method for finding the solution, i.e. calculation of the "successive approximations" ($x_n = A^n x_0$). In fact, as shown in the proof, the approximations actually converge to the solution of the equation $Ax = x$. For this reason, the fixed point theorem if often called the method of successive approximations.

Example 68.1. Let $f$ be a function defined on the closed interval $[a, b]$ which maps $[a, b]$ into itself and satisfies a Lipschitz condition

(1) $$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

with constant $K < 1$. Then $f$ is a contraction map, and hence, by Theorem (68.1), the sequence

(2) $$x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots$$

converges to the unique root of the equation $f(x) = x$. In particular, "contraction condition" (1) holds if $f$ has a continuous derivative $f'$ on $[a, b]$ such that

$$|f'(x)| \leq K < 1$$

Example 68.2. Consider the mapping $A$ of $n$-dimensional space into itself given by the system of linear equations

(3) $$y_i = \sum_{j=1}^{n} a_{ij}x_j + b_i \quad (i = 1, \ldots, n)$$

If $A$ is a contraction mapping, we can use the method of successive approximations to solve the equation $Ax = x$. The condition under which $A$ is a contraction mapping depend on the choice of metric. We now examine three cases:

1. The space $R^n_0$ with metric

$$\rho(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

In this case,

$$\rho(y, \tilde{y}) = \max_i |y_i - \tilde{y}_i| = \max_i \left| \sum_j a_{ij}(x_j - \tilde{x}_j) \right|$$

$$\leq \max_i \sum_j |a_{ij}| |x_j - \tilde{x}_j|$$

$$\leq \max_i \sum_j |a_{ij}| \max_j |(x_j - \tilde{x}_j)| = (\max_j \sum_i |a_{ij}|) \rho(x, \tilde{x})$$
and the contraction condition is now

\[ \sum_{j} |a_{ij}| \leq \alpha < 1 \ (j = 1, \ldots, n) \]  

2. The space \( R^n_1 \) with metric

\[ \rho(x, y) = \sum_{i=1}^{n} |x_i - y_i| \]

Here

\[ \rho(y, \tilde{y}) = \sum_{i} |x_i - \tilde{y}_i| = \sum_{i} \left| \sum_{j} a_{ij}(x_j - \tilde{x}_j) \right| \]
\[ \leq \sum_{i} \sum_{j} |a_{ij}| |(x_j - \tilde{x}_j)| \]
\[ \leq (\max_{i} \sum_{j} |a_{ij}|) \rho(x, \tilde{x}) \]

and the contraction condition is now

\[ \sum_{j} |a_{ij}| \leq \alpha < 1 \ (j = 1, \ldots, n) \]

3. Ordinary Euclidean space \( R^n \) with metric

\[ \rho(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \]

Using the Cauchy-Schwarz inequality, we have

\[ \rho^2(y, \tilde{y}) = \sum_{i} \left( \sum_{j} a_{ij}(x_j - \tilde{x}_j) \right)^2 \leq \left( \sum_{i} \sum_{j} a_{ij}^2 \right) \rho^2(x, \tilde{x}) \]

and the contraction condition becomes

\[ \sum_{i} \sum_{j} a_{ij}^2 \leq \alpha < 1 \]
Thus, if at least one of conditions (4-6) holds, there exist a unique point \( x = (x_1, x_2, \ldots, x_n) \) such that

\[
x_i = \sum_{i=1}^{n} a_{ij} x_j + b_i \quad (i = 1, \ldots, n)
\]

The sequence of successive approximations to this solution of the equation \( x = Ax \) are of the form

\[
x^0 = (x_0^0, x_0^1, \ldots, x_0^n) \\
x^1 = (x_1^0, x_1^1, \ldots, x_1^n) \\
x^k = (x_k^0, x_k^1, \ldots, x_k^n)
\]

where

\[
x_i^k = \sum_{i=1}^{n} a_{ij} x_{j}^{k-1} + b_i
\]

and we can choose any point \( x^0 \) as the "zeroth approximation".

Each of the conditions (4-6) is sufficient for applicability of the method of successive approximations, but none of them is necessary. In fact, examples can be constructed in which each of the conditions (4-6) is satisfied, but not the other two.

### 68.2 Contraction mapping and differential equations

The most interesting applications of Theorem (68.1) arise when the space \( R \) is a function space.

We can use this theorem to prove a number of existence and uniqueness theorems for differential and integral equations.

**Theorem 68.2. (Picard).** Given a function \( f(x,y) \) defined and continuous on a plane domain \( G \) containing the point \((x_0, y_0)\) suppose \( f \) satisfies a Lipschitz condition of the form

\[
|f(x,y) - f(x,\tilde{y})| \leq M|y - \tilde{y}|
\]

in the variable \( y \). Then there is an interval \( |x - x_0| \leq \delta \) in which the differential equation

\[
\frac{dy}{dx} = f(x,y)
\]

has a unique solution

\[
y = \varphi(x)
\]

satisfying the initial condition

\[
\varphi(x_0) = y_0
\]
Remark. By an $n$–dimensional domain we mean an open connected set in Euclidean $n$–space.

Proof. Together the differential equation (7) and the initial condition (8) are equivalent to the integral equation

$$(9) \quad \varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t)) dt$$

By the continuity of $f$, we have

$$(10) \quad |f(x, y)| \leq K$$

in some domain $G' \subset G$ (in fact $f$ is bounded on $G' \subset G$) containing the point $(x_0, y_0)$. Choose $\delta > 0$ such that

1. $(x, y) \in G'$ if $|x - x_0| \leq \delta$, $|y - y_0| \leq K\delta$
2. $M\delta < 1$

and let $C^*$ be the space of continuous functions $\varphi$ defined on the interval $|x - x_0| \leq \delta$ and such that $|\varphi(x) - y_0| \leq K\delta$, equipped with the metric

$$\rho(\varphi, \tilde{\varphi}) = \max_{x} |\varphi(x) - \tilde{\varphi}(x)|$$

The space $C^*$ is complete, since it is closed subspace of the space of all continuous functions on $[x_0 - \delta, x_0 + \delta]$. Consider the mapping $\psi = A\varphi$ defined by the integral equation

$$\psi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t)) dt \ (|x - x_0| \leq \delta)$$

Clearly $A$ is a contraction mapping carrying $C^*$ into itself. In fact, if $\varphi \in C^*$, $|x - x_0| \leq \delta$ then

$$|\psi(x) - y_0| = |\int_{x_0}^{x} f(t, \varphi(t)) dt| \leq$$

$$\leq \int_{x_0}^{x} |f(t, \varphi(t))| dt \leq$$

$$\leq K|x - x_0| \leq K\delta$$

by (10), and hence $\psi = A\varphi$ also belongs to $C^*$. Moreover,

$$|\psi(x) - \tilde{\psi}(x)| \leq \int_{x_0}^{x} |f(t, \varphi(t)) - f(t, \tilde{\varphi}(t))| dt$$

$$\leq M\delta|\varphi(t) - \tilde{\varphi}(t)|$$
and hence
\[ \rho(\psi, \tilde{\psi}) \leq M\delta \rho(\psi, \tilde{\psi}) \]

after maximizing with respect to \( x \). But \( M\delta < 1 \), so that \( A \) is a contraction mapping. It follows from Theorem (68.1) that the equation \( \varphi = A\varphi \), i.e. the integral equation (9), has a unique solution in the space \( C^* \). \( \square \)