

1 [25 pts.] Find general solution to the equation $2tdy + (y - 1)dt = 0$.

$$\begin{aligned} 2t \, dy &= (1-y) \, dt \\ \int \frac{dy}{1-y} &= \sqrt{\frac{dt}{2t}} \\ -\ln(1-y) &= \frac{1}{2} \ln t + C \\ \ln(1-y) &= \ln t^{-\frac{1}{2}} + C \\ 1-y &= e^{\ln t^{-\frac{1}{2}} + C} \\ 1-y &= Ct^{-\frac{1}{2}} \\ y &= 1-Ct^{-\frac{1}{2}} \end{aligned}$$

2 [25 pts.] Solve the initial value problem

$$y' + 3x^{-1}y = 5x, \quad y(1) = 3.$$

$$\begin{aligned} M(x) &= e^{\int \frac{3}{x} dx} = e^{3\ln x} = x^3 \\ \int [x^3 y]' dx &= \int 5x^4 dx \\ x^3 y &= x^5 + C \\ y &= x^2 + Cx^{-3} \\ 3 &= 1 + C(1) \\ C &= 2 \\ y &= x^2 + 2x^{-3} \end{aligned}$$

3 [25 pts] Solve $(e^{-x}y - xe^{-x}y)dx + (xe^{-x} + 2)dy = 0$, $y(0) = 1$.

$$M = e^{-x}y - xe^{-x}y, N = xe^{-x} + 2$$

$$\frac{\partial M}{\partial y} = e^{-x} - xe^{-x} = \frac{\partial N}{\partial x} \Rightarrow \text{exact eqn.}$$

$$F = \int M dx = -ye^{-x} + y \int x d(e^{-x}) = -ye^{-x} + xy e^{-x} - y \int e^{-x} dx \\ = xy e^{-x} + C(y)$$

$$\frac{\partial F}{\partial y} = xe^{-x} + C'(y) = N = xe^{-x} + 2 \Rightarrow C(y) = 2y.$$

$$F = xy e^{-x} + 2y = C$$

$$y(0) = 1 \Rightarrow 2 \cdot 1 = C \Rightarrow C = 2$$

$$(xe^{-x} + 2)y = 2 \quad \text{Answer: } y = \frac{2}{xe^{-x} + 2}$$

4 [25pts] Solve $y' - 4y = 2xy^2$, $y(0) = 2$.

$$v = \frac{1}{y}, y = \frac{1}{v}; y' = -\frac{v'}{v^2} \Rightarrow -\frac{v'}{v^2} - \frac{4}{v} = \frac{2x}{v^2}$$

$$v' + 4v = -2x$$

$$M = e^{4x}, (e^{4x}v)' = -2 \int xe^{4x} dx = -\frac{1}{2} \int x d(e^{4x})$$

$$= -\frac{1}{2} xe^{4x} + \frac{1}{2} \int e^{4x} dx$$

$$= -\frac{1}{2} xe^{4x} + \frac{1}{8} e^{4x} + C$$

$$v(x) = -\frac{1}{2}x + \frac{1}{8} + Ce^{-4x}$$

$$v(0) = \frac{1}{2} \Rightarrow C = \frac{3}{8}.$$

3

$$\text{Answer: } y(x) = \frac{1}{-\frac{1}{2}x + \frac{1}{8} + \frac{3}{8}e^{-4x}} = \frac{8}{3e^{4x} + 1 - 4x}$$