Ma 635. Real Analysis I. Hw2 (due 09/14). Solutions.

1. [2] p. 45 # 1

Given a metric space (X, d), prove that
(a) $|d(x,z) - d(y,u)| \leq d(x,y) + d(z,u)$
(b) $|d(x,z) - d(y,z)| \leq d(x,y)$

Solution. (a) The triangle inequality yields: $d(x,z) \leq d(x,y) + d(y,u) + d(u,z)$. Then, $d(x,z) - d(y,u) \leq d(x,y) + d(u,z)$. Similarly, $d(y,u) - d(x,z) \leq d(x,y) + d(u,z)$. The result follows from the last two inequalities. (b) can be obtained similarly.

2. [2] p. 45 # 5

Prove that the metric in $(-\infty, +\infty)$, $d_\infty(x,y) = \max\limits_{1\leq k\leq n} |x_k - y_k|$ is the limiting case of the metric $d_p(x,y) = \left(\sum\limits_{k=1}^{n} |x_k - y_k|^p\right)^{1/p}$ as $p \to \infty$.

Solution. Let $\max\limits_{1\leq k\leq n} |x_k - y_k|$ be attained at $k = k_0$: $\max\limits_{1\leq k\leq n} |x_k - y_k| = |x_{k_0} - y_{k_0}|$. Then

$$\begin{aligned}\left(\sum_{k=1}^{n} |x_k - y_k|^p\right)^{1/p} &= |x_{k_0} - y_{k_0}| \left(\sum_{k=1}^{n} \left|\frac{x_k - y_k}{x_{k_0} - y_{k_0}}\right|^p\right)^{1/p} \\ &\leq |x_{k_0} - y_{k_0}| n^{1/p} \to |x_{k_0} - y_{k_0}|.\end{aligned}$$

From another point of view,

$$\left(\sum_{k=1}^{n} |x_k - y_k|^p\right)^{1/p} \geq \left(|x_{k_0} - y_{k_0}|^p\right)^{1/p} = |x_{k_0} - y_{k_0}|.$$

From two last expressions we arrive at the solution.

3. [2] p. 45 # 8

Exhibit an isometry between the spaces $C[0,1]$ and $C[1,2]$.

Solution. Let $f : C[0,1] \mapsto C[1,2]$ as follows: $f(x)(t) = x(t+1)$. Clearly,

$$d(f(x), f(y)) = \sup_{1\leq t\leq 2} |f(x)(t) - f(y)(t)| = \sup_{0\leq t\leq 1} |x(t) - y(t)| = d(x,y).$$

Hence, f is an isometry.

4. [2] p. 54 # 3

Prove that if $x_n \to x$, $y_n \to y$ as $n \to \infty$ then $d(x_n, y_n) \to d(x,y)$.

Solution. From the corollary from triangle inequality, $|d(x_n, y_n) - d(x,y)| \leq d(x_n, x) + d(y_n, y) \to 0$.

5. [2] p. 54 # 7

Solution. In the ternary number system, $\left(\frac{1}{4}\right)_{10} = \left(\frac{1}{11}\right)_3 = 0.02020202...$ which has no ones. Since in the ternary system the elements of the Cantor discontinuum and only they are presented by ternary fractions without ones, then 1/4 belongs to the Cantor set.

6. [2] p. 65 # 2

Prove that space $m = l_\infty$ of bounded sequences with metric $d(x,y) = \sup\limits_{1\leq k\leq\infty} |x_k - y_k|$ is complete.

Solution. Let $\left\{x^{(n)}\right\} \subset l_\infty$ be a Cauchy sequence. Since for sufficiently large n and m, $d(x^{(n)}, x^{(m)}) = \sup\limits_{1\leq k\leq\infty} |x_k^{(n)} - x_k^{(m)}| < \varepsilon$, then the number sequence $x_1^{(n)}$ is a Cauchy sequence with limit x_1, the number sequence $x_2^{(n)}$ is a Cauchy sequence with limit x_2, and so on. So, we arrive at the sequence $\{x_k\}$, which is a coordinatewise limit of $\left\{x^{(n)}\right\}$. Now we should show only that $\{x_k\} \in l_\infty$. But it follows directly from the theorem on the boundedness of any Cauchy sequence.

7. [2] p. 65 # 4

Suppose metric space R is complete, and let $\{A_n\}$ be a sequence of closed subsets of R nested in the sense that

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

Let also the diameters tend to zero: $\lim_{n\to\infty} d(A_n) = 0$. Prove that the intersection $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

Solution. We construct a Cauchy sequence as follows. Let $x_1 \in A_1 \setminus A_2$, $x_2 \in A_2 \setminus A_3$, and so on. Since $\forall \varepsilon > 0\ \exists N$ such that $\forall\, n > N$, $d(A_n) < \varepsilon$ then also $d(x_n, x_m) < \varepsilon$ if $n, m > N$. Hence, $\{x_n\}$ is really a Cauchy sequence. Let x be its limit. Since A_1 is closed and $\{x_n\}_{n=1}^{\infty} \subset A_1$ then $x \in A_1$ (closed sets contain the limits of their convergent subsequences). Since A_2 is closed and $\{x_n\}_{n=2}^{\infty} \subset A_2$ then $x \in A_2$, and so on. Consequently, $\forall n\, x \in A_n$, or $x \in \bigcap_{n=1}^{\infty} A_n$.

8. [1] p. 38 # 1

Show that

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

defines a metric on $(0, \infty)$.

Solution. Positivity and symmetry of d are obvious. Also, $d(x,x) = 0$. To see the validity of the triangle inequality, we observe that

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| \le \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right| = d(x,z) + d(y,z).$$

9. [1] p. 38 # 6

If d is any metric on M, show that $\rho(x,y) = \sqrt{d(x,y)}$, $\sigma(x,y) = \frac{d(x,y}{1+d(x,y)}$, and $\tau(x,y) = \min\{d(x,y), 1\}$ are also metrics on M.

Solution. If $F(0) = 0$ and $F(\alpha) > 0$ for $\alpha > 0$ then clearly $\theta = F(d)$ satisfies all the properties of metric but the triangle inequality. To find the additional conditions to be imposed on F, let us consider when $\theta(x,y) \le \theta(x,z) + \theta(y,z)$ in view of the inequality $d(x,y) \le d(x,z) + d(y,z)$:

$$\theta(x,y) = F(d(x,y)) \le F(d(x,z) + d(y,z)) \le F(d(x,z)) + F(d(y,z)).$$

The first inequality holds if (1) F in non-decreasing. The second inequality holds if (2) $\forall \alpha, \beta \ge 0$, $F(\alpha + \beta) \le F(\alpha) + F(\beta)$. All the above functions ρ, σ, τ satisfy these two conditions. Consequently, they are metrics.

10. [1] p. 39 # 11

Let R^∞ be the space of all infinite dimensional vectors $\{x_n\}_{n=1}^{\infty}$. Show that the expression

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

defines a metric on R^∞.

Solution. Use the previous solution for σ applied independently to every coordinate. The series converges due to the coefficient $1/n!$.

11. [1] p. 39 # 12

Check that $d(x,y) = \sup_{a \le t \le b} |x(t) - y(t)|$ defines a metric on $C[a,b]$, the space of all continuous functions defined on the closed interval $[a,b]$.

Solution.

$$\begin{aligned} d(x,y) &= \sup_{a\le t\le b} |x(t) - y(t)| \le \sup_{a\le t\le b} (|x(t) - z(t)| + |z(t) - y(t)| \\ &\le \sup_{a\le t\le b} |x(t) - z(t)| + \sup_{a\le t\le b} |y(t) - z(t)| = d(x,z) + d(y,z). \end{aligned}$$

All other properties of metric are obvious.

12. [1] p. 42 # 23
The subset of l_∞ consisting of all sequences that converge to 0 is denoted by c_0. Show that we have the following proper set inclusions: $l_1 \subset l_2 \subset c_0 \subset l_\infty$.

Solution. If $x \in l_1$ then $\sum\limits_{n=1}^{\infty} |x_n| < \infty$, then $\sum\limits_{n=1}^{\infty} |x_n|^2 < \infty$, then $x_n \to 0$, then x_n is a bounded sequence. Consequently, $l_1 \subseteq l_2 \subseteq c_0 \subseteq l_\infty$.

The inclusion is proper. Really, $(1,1,1,\ldots) \in l_\infty \setminus c_0$,
$\{\frac{1}{\sqrt{n}}\} \in c_0 \setminus l_2$,
$\{\frac{1}{n}\} \in l_2 \setminus l_1$.

13. [1] p. 46 # 34
If $x_n \to x$ in (M,d), show that $\forall y \in M,\ d(x_n, y) \to d(x,y)$.
Solution. Look # 4 above.

14. [1] p. 46 # 37
A Cauchy sequence with a convergent subsequence converges.
Solution. Let x be the limit of the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.
Then $\forall \varepsilon > 0\ \exists K\ \forall k > K,\ d(x, x_{n_k}) < \varepsilon/2$.
Since this is a Cauchy sequence then $\exists N = N(\varepsilon),\ \forall n, m > N,\ d(x_n, x_m) < \varepsilon/2$.
Let $M = \max\{K, N\}$. Since $n_k \geq k$ then $n_k, m > M$, and $d(x, x_m) \leq d(x, x_{n_k}) + d(x_{n_k}, x_m) < \varepsilon$, which implies the convergence of the Cauchy sequence to x.

(bonus 1) [2] p. 53 #1
Give an example of a metric space R and two open balls $B_{r_1}(x)$ and $B_{r_2}(x)$ in R such that $B_{r_1}(x) \subset B_{r_2}(y)$ although $r_1 > r_2$.
Solution (Vitalii K.) As an example consider metric space $R = \{x, y, z\}$, composed of three points x, y, z. Let $z \in B_{r_2}(y)$ and $z \notin B_{r_1}(x)$, $r_2 < r_1 \Rightarrow \rho(y,z) < r_2 < r_1$, $\rho(x,z) > r_1 > r_2$, $\rho(x,y) < r_2 < r_1$. From triangle inequality

$$r_1 < \rho(x,z) \leq \rho(x,y) + \rho(y,z) < 2r_2 \tag{1}$$

The triangle inequality determines the condition of strict inclusion $B_{r_1}(x) \subset B_{r_2}(y)$. Thus, let us assume $r_1 = 6$, $r_2 = 5$, $\rho(x,y) = \rho(y,z) = 4$, $\rho(x,z) = 7$. Then $B_{r_1}(x) = \{x, y\}$, $B_{r_2}(y) = \{x, y, z\}$. Then $B_{r_1}(x) \subset B_{r_2}(y)$. Graphically we can interpret this situation as triangle with x, y, z in its vertexes and sides $\rho(x,y) = \rho(y,z) = 4$, $\rho(x,z) = 7$.

(bonus 2) [2] p. 65 # 6
Give an example of a complete metric space R and a nested sequence $\{A_n\}$ of closed subsets of R such that

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Reconcile this example with Problem 4.
Solution 1. Consider $R = \{1, 2, 3, \ldots\}$ with metric $d(x,y) = 1 + \frac{1}{x+y}$ for distinct x, y. This space is complete: all Cauchy sequences have stationary "tails". In this space
$\overline{B}_{4/3}(1) = \{1, 2, 3, 4, 5, \ldots\}$,
$\overline{B}_{6/5}(2) = \{2, 3, 4, 5, \ldots\}$,
$\overline{B}_{8/7}(3) = \{3, 4, 5, \ldots\}$,....
Consequently,

$$\overline{B}_{4/3}(1) \supset \overline{B}_{6/5}(2) \supset \overline{B}_{8/7}(3) \supset \ldots \supset \overline{B}_{(2n+2)/(2n+1)}(n) \supset \ldots.$$

However, $\bigcap\limits_{n=1}^{\infty} \overline{B}_{(2n+2)/(2n+1)}(n) = \emptyset$.

Solution 2. (Vitalii K.) As an example consider metric space $\mathbf{R} = (-\infty, +\infty)$. Consider $A_n = (-\infty, a_n] \subset \mathbf{R}$, $a_n \to -\infty$ (for example, $a_n = (1, 0, -2, -3\ldots)$. Then $\{A_n\}$ is a nested sequence of closed subsets of $\mathbf{R}$:

$$A_1 \supset A_2 \supset A_3 \supset \ldots \supset A_n \supset \ldots$$

Obviously,

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Reconciling this example with Problem 4 we see that in the suggested example $d(A_n) = \infty$ for all n whereas in Problem 4 assume

$$\lim_{n\to\infty} d(A_n) = 0.$$

Remark: unlike the first solution, the second one deals with unbounded sets, which is easier.

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