Ma 635. Real Analysis I. Hw5, due 10/05.

HW 5. (due 10/05). Solutions.

1. [1] p. 92 # 9

Give an example of a closed bounded subset of l_∞ that is not totally bounded.
Solution. Just the set of unit coordinate vectors like $(0,0,1,0,0,\ldots)$.

2. [1] p. 110 # 12

Show that the set $A = \{x \in l_2 : \ |x_n| \le \frac{1}{n}\}$ is compact in l_2.
Done in class, compare with the next problem.

3. [1] p. 110 # 14

Show that the Hilbert cube $H^\infty = \{x = (x_n)_{n=1}^\infty, \ |x_n| \le 1\}$ is compact if

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

Solution. $\forall\, \varepsilon$ we first choose N such that $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon/2$ (for example, $N > \log_2 \frac{1}{\varepsilon}$). Then for every coordinate from 1 to N we choose $2N+1$ grid points $\pm\frac{k\varepsilon}{2N}$ that are distant by $\varepsilon/(2N)$ from each other, $-N \le k \le N$. To form the ε-net, we pick up all possible grid points for first N coordinates, and zeros for the remaining infinite "tail".

4. [1] p. 110 # 17

If M is compact, show that M is also separable.
Solution. In view of total boundedness, for any $\varepsilon_n = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ the exists a bounded ε_n-net. The union of these ε_n-nets is just a dense countable subset.

5. [1] p. 110 # 18

M has a countable open base if and only if M is separable.
Solution. ($\Longrightarrow$) If M has a countable open base, we choose a point inside each element of the base. These points form a dense and separable subset.
($\Longleftarrow$) If M is separable then consider the collection of open balls with rational radiuses centered at the elements of the dense countable subset of M.

6. [1] p. 111 # 23

If M is compact and $f : \ M \mapsto N$ is a continuous bijection then M is homeomorphism.
Solution. We need to show only that f^{-1} is also continuous. Let $y_n \to y$ in N. Consider x_n - their pre-images in M, $x_n = f^{-1}(y_n)$, $x = f^{-1}(y)$. Since M is compact then sequence $\{x_n\}$ contains a converging subsequence $\{x_{n_k}$ with limit, say, z. Since f is continuous then $f(x_{n_k}) \to f(z)$, or $f(y_{n_k}) \to f(z)$. However, since $y_n \to y$ then also $y_{n_k} \to y$ and, hence, $f(z) = y$. Then $z = f^{-1}(y) = x$. Consequently, $x_{n_k} \to x$. Consider the terms of $\{x_n\}$ outside the subsequence $\{x_{n_k}\}$. Because of the above argument, those remaining terms cannot have a subsequence that converges to a value other than x. Since all converging subsequences of $\{x_n\}$ converge to x, then $x_n \to x$, which proves the continuity of f^{-1}.

7. [1] p. 114 # 34

A is closed in $M \iff A \cap K$ is compact $\forall\, K$-compact.
Solution. ($\Longrightarrow$) As a subset of K, $A \cap K$ is totally bounded. As an intersection of two closed sets, $A \cap K$ is closed. Then it is compact.
($\Longleftarrow$) We consider a Cauchy sequence $\{x_n\} \subset A$. Let $K = \{x_n\} \cup x$ where x is the limit of x_n. Since $A \cap K$ is compact then $x \in A$.

8 [1] p. 114 # 35

Every open cover $\mathcal{G}$ of compactum M has a Lebesgue number $L(\mathcal{G}) > 0$. By definition,

$$L(\mathcal{G}) = \inf_{\varepsilon>0}\{\forall B_\varepsilon \subset M\ \exists G \in \mathcal{G} :\ B_\varepsilon \subseteq G\}.$$

Solution. If a cover $\mathcal{G}$ has no positive Lebesgue number, then it is equal to zero. Then $\forall\ \{\varepsilon_n\}_{n=1}^{\infty}$, $\varepsilon_n \to 0$, $\exists B_{\varepsilon_n} \subset M$ such that $\forall G \in \mathcal{G},\ B_{\varepsilon_n} \not\subset G$. Let $\{G_n\}_{n=1}^{N}$ be a finite open subcover of $\mathcal{G}$ that covers M. We consider the open subsets $D_n = B_{\varepsilon_n} \setminus G_n$. They are non-empty (otherwise $B_{\varepsilon_n} \subset G_n$). Their intersection is also non-empty (why?). Then $\bigcap_{n=1}^{N} D_n \not\subset M$, which contradicts the inclusion $D_n \subseteq M$ for all n.

9 [1] p. 114 # 36

10 [2], p. 81, # 4.
Show that $E = \{1/n :\ n$ a positive integer $\}$ is not compact in $\mathbf{R}$ but $E \cup \{0\}$ is compact.

11 [2], p. 81, # 7.
Prove that every finite subset of a metric space is compact.
Solution. Any sequence from a finite subset contains a stationary subsequence, which is convergent.

12 [2], p. 81, # 6.
Show that a discrete metric space M is not compact unless X is finite.
Solution. Let x_n be a countable infinite sequence in M. Since the distance between any two points is equal to 1, this sequence has no converging subsequence and, thus, is not compact.

13 [2], p. 81, # 9
If X is compact prove that $C(X, \mathbf{R})$ is a complete metric space.
Slightly modify the theorem from Advanced Calculus that states that the uniform convergence of continuous functions yield a continuous function as the pointwise limit.

14 [2], p. 81, # 10
Is $C[0, 1]$ compact?
No, $x_n(t) = t^n$ is a not relatively compact subset, its pointwise limit is a discontinuous function.

15. [2], p. 84, # 4 Prove that any compact metric space has a dense countable subset.
See # 4.

16. [3], p. 115, # 5
Let X be a metric compactum and $A :\ X \mapsto X$ such that $d(Ax,\ Ay) < d(x, y)$ if $x \neq y$. Prove that A has a unique fixed point.
Solution. Consider arbitrary x_0 and let $x_n = Ax_0$. Let $d_1 = d(x_0, x_1)$, $d_2 = d(x_1, x_2), \ldots$. As we know, $d_1 > d_2 > d_3 > \cdots$. Since X is compact, we pick up x_{n_k} - a converging subsequence of $\{x_n\}$, $x_{n_k} \to z_0$ as $k \to \infty$. That implies $d(x_{n_k}, x_{n_m}) \to 0$ as $k, m \to \infty$. We consider the sequence $Ax_{n_k} = x_{n_k+1}$. From the problem statement, $d(Ax_{n_k}, Ax_{n_m}) < d(x_{n_k}, x_{n_m}) \to 0$. Hence, the sequence Ax_{n_k+1} is also a Cauchy sequence, let its limit be z_1. Similarly, we obtain $\exists\ \lim x_{n_k+2} = z_2$, $\exists\ \lim x_{n_k+3} = z_3$, and so on. The number of limit points $z_0, z_1, z_2, \ldots$ cannot exceed $n_{k+1} - n_k$ since the sequence $x_{n_{k+1}}$ is just x_{n_k} and, hence, $x_{n_{k+1}} \to z_0$. So, we have limit points $z_0, z_1, \ldots, z_p$. The condition $d(Ax,\ Ay) < d(x, y)$ implies that A is a continuous mapping. Since $x_{n_k} \to z_0$ then $Ax_{n_k} \to Az_0$. Consequently, $z_1 = Az_0$. Similarly, $z_2 = Az_1$, $z_3 = Az_2$, $z_0 = Az_p$. However, the condition $d(Ax,\ Ay) < d(x, y)$ implies that there may not be a periodic orbit. Really,

$$d(z_1, z_0) = d(Az_0, Az_p) < d(z_0, z_p) = d(Az_p, Az_{p-1}) < \cdots < d(z_2, z_1) = d(Az_1, Az_0) < d(z_1, z_0).$$

The contradiction $d(z_1, z_0) < d(z_1, z_0)$ proves the absence of any periodic orbit. So, $z_0 = z_1 = \ldots = z_p$ is the only fixed point: $z_0 = Az_0$.

17. Prove that a uniformly bounded set of functions in $C[a, b]$, which satisfy the Lipshitz condition with the same common constant, is compact in $C[a, b]$.
$x(t)$ satisfies the Lipshitz condition with constant L if $\forall t, s :\ |x(t) - x(s)| \leq C|t - s|$.

18. Determine whether the following sets in $C[0,1]$ are relatively compact (pre-compact):
(a) $x_n(t) = \sin(nt)$
Not relatively compact: $[\sin(nt)]' = n\cos(nt) = n \to \infty$ at $t = 0$.

(b) $x_n(t) = \sin(t+n)$
This set is pre-compact: $|x_n'(t)| = |\cos(t+n)| \leq 1$. Uniform boundedness of slopes implies equicontinuity.

(c) $x_\alpha(t) = \arctan(\alpha t)$, $\alpha \in \mathbf{R}$
Not pre-compact. Do like in (a)

(d) $x_\alpha(t) = e^{t-\alpha}$, $\alpha \in \mathbf{R}$, $\alpha \geq 0$.
Pre-compact. $|x_\alpha'(t)| = e^{t-\alpha} \leq e$ since $0 \leq t \leq 1$.

Bonus 2: Prove that the condition $d(f(x), f(y)) < d(x,y),\ x \neq y$, is insufficient for the existence of a fixed point of function f.
Solution. Consider in usual metric $f:\ [0,\infty) \mapsto [0,\infty)$, $f(x) = x + \frac{1}{x+1}$. The graph $y = f(x)$ is above the bisector $y = x$ and approaches it as $x \to \infty$. Its slope is positive but less than 1 at all points. Hence, $d(f(x), f(y)) < d(x,y)$.

Bonus 3; [1] p. 111 # 24.

References

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