Ma 635. Real Analysis I. Review I.

1. Prove that the function $f(t) = t^3$ is a contractive operator in the ball $B_r(0)$, $r < 1/\sqrt{3}$, but isn't contractive near the fixed points $t = 1$ and $t = -1$.

Solution. $|f(t) - f(s)| = |t^2 + ts + s^2| \cdot |t - s|$. So, $|t^2 + ts + s^2| \leq \alpha < 1$ if $|t|, |s| < \frac{1}{\sqrt{3}}$. Obviously, $f : B_r(0) \mapsto B_r(0)$, $r < \frac{1}{\sqrt{3}}$ (invariance). Consequently, in view of the contraction mapping theorem, $\exists! t \in B_r(0)$ such that $t = f(t)$. The uniqueness holds in $B_r(0)$ only.
Near the fixed points ± 1 the expression $|t^2 + ts + s^2| \approx 3 > 1$, and we have no contraction then.

2. Let us consider the set of n-dimensional vectors $x = \big\langle x_1, x_2, \ldots x_n \big\rangle$. Find $0 < p < 1$ such that $\|x\| = \left[\sum_{i=1}^n |x_i|^p\right]^{1/p}$ is not a norm.

Solution: Let $p = 1/2$, $n = 2$, and $x_i,\ y_i > 0$. Then

$$\|x\| = x_1 + x_2 + 2\sqrt{x_1 x_2}, \qquad \|y\| = y_1 + y_2 + 2\sqrt{y_1 y_2},$$

$$\|x + y\| = \left(\sqrt{x_1 + y_1} + \sqrt{x_2 + y_2}\right)^2 = x_1 + x_2 + y_1 + y_2 + 2\sqrt{(x_1 + y_1)(x_2 + y_2)} > \|x\| + \|y\|$$

because $\sqrt{(x_1 + y_1)(x_2 + y_2)} > \sqrt{x_1 x_2} + \sqrt{y_1 y_2}$.
To prove the last inequality, we square it and obtain

$$\frac{x_1 y_2 + x_2 y_1}{2} > \sqrt{x_1 y_2 \cdot x_2 y_1},$$

which is just the classical arithmetic-geometric inequality $\frac{a+b}{2} \geq \sqrt{ab}$.
Consequently, the triangle inequality fails.

3. Let c_0 be the space of infinite vectors (sequences) $x = (x_1, x_2, \ldots, x_n, \ldots)$, converging to zero: $\lim\limits_{n \to \infty} x_n = 0$ with a metric $d(x, y) = \sup\limits_n |x_n - y_n|$.
Prove that
(a) $\forall p \geq 1, \quad l_p \subset c_0$
(b) $\forall p \geq 1, \quad c_0 \not\subset l_p$.

Solution (a) If $x \in l^p$ then $\sum_{i=1}^\infty |x_i|^p < \infty$ and, consequently, $x_i \to 0$ as $i \to \infty$. Thus, $x \in c_0$.

(b) Consider, say,

$$x = \left(1, \left(\frac{1}{2}\right)^{1/p}, \left(\frac{1}{3}\right)^{1/p}, \left(\frac{1}{4}\right)^{1/p}, \ldots, \left(\frac{1}{i}\right)^{1/p}, \ldots\right) \in c_0.$$

However, $x \notin l_p$ since the series $\sum_{i=1}^\infty \frac{1}{i}$ diverges.

4. Whether the sequence $x_n(t) = \dfrac{t^{2n}}{2n} - \dfrac{t^{3n}}{3n}$ is converging in space
(a) $C[0, 1]$
(b) $C^1[0, 1]$, where $\|x\|_{C_1} = \|x\|_C + \|x'\|_C$
(c) $C^2[0, 1]$, where $\|x\|_{C_1} = \|x\|_C + \|x'\|_C + \|x''\|_C$
(d) $C[0, \frac{1}{2}]$?

Solution Since the sequence converges pointwise (for every fixed $t \in [0, 1]$ to zero, then we should verify the convergence of the given sequence to zero in corresponding spaces.

(a) $\|x_n - 0\| = \sup\limits_{0 \leq t \leq 1} |\frac{t^{2n}}{2n} - \frac{t^{3n}}{3n}| = \frac{1}{6n} \to 0$. Hence, the sequence is converging in $C[0, 1]$.

(b) $\|x_n\|_{C^1} = \|x_n\|_C + \|x_n'\|_C.\|x_n'\|_C = \sup\limits_{0 \leq t \leq 1} |t^{2n-1} - t^{3n-1}|$. To evaluate the maximum, we differentiate and equal the derivative to zero. Afterwards, we obtain $\|x_n'\| \to frac427 \neq 0$. No convergence in $c^1[0, 1]$.

(c) Clearly, no convergence, see (b).
(d) In view of (a), it is convergent in $C[0, \frac{1}{2}]$.

5. Whether the set of all polynomials in $C[a,b]$
(a) is open? (b) is closed?

Solution. Neither open, nor closed. Let $P \subset C[a,b]$ be the set of all polynomials defined on $[a,b]$. If P is open then $\forall x \in P \exists B_\varepsilon(x) \subset P$. However, $y_\varepsilon(t) = 1 + \frac{\varepsilon}{2}\sin t \in B_\varepsilon(1)$ but $y_\varepsilon \not{i}nP$.

If P is closed, then any convergent sequence of polynomials must have a polynomial as a limit. However, the sequence $p_n(t) = \sum\limits_{i=0}^{n} \frac{t^n}{n!}$ converges to the sum of the infinite series, which is just $e^t \notin P$.

6. For which values p the set $\{x \in l_p,\ x = (x_1, x_2, \ldots) : \ |x_n| \le 1/n\}$ is closed?

For all $p \ge 1$.

7. It is known that the set of continuous piecewise linear functions is dense in $C[a,b]$. Is $C[a,b]$ separable?

Yes, let us consider continuous piecewise linear functions with cusps at rational points. The number of such functions is countable, and they approximate any continuous piecewise linear functions, which, in turn, approximate any continuous functions.

8. (a) Is the function $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ a metric on the set of all real numbers $x \ge 1$?

(b) If "yes", is this metric space complete?

No.

(c) If it is not complete, describe its completion (what points should be "added"?).

Add two "infinite" points $\pm\infty$.

9. Prove that every finite-dimensional linear normed space is complete.

All norms in finite-dimensional spaces are equivalent:

$$\exists c_1, c_2 \text{ such that } \forall x \ c_1\|x\|_1 \le \|x\| \le c_2\|x\|_1.$$

So it is sufficient to consider the standard norm in R^m. Any Cauchy sequence in m-dimensional space has a coordinate-wise limit, which is just the desired limit in R^m. Really, the sequence of the first coordinates is a Cauchy sequence in the standard sense of number sequences and, hence, it has a limit. The same is true for any coordinate. The limits of coordinates form the m-dimensional vector, which is just the coordinate-wise limit of the original Cauchy sequence.

10. Consider space R^∞ of all sequences $x = (x_1, x_2, \ldots)$.

(a) Show that

$$d(x,y) = \sum_{i=1}^{\infty} 2^{-i}\frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

is a metric.

Check the properties of metric, especially the triangle inequality.

(b) Prove that R^∞ is complete.

$d\left(x^{(n)}, x^{(m)}\right) = \sum_{i=1}^{\infty} 2^{-i}\frac{|x_i^{(n)} - x_i^{(m)}|}{1+|x_i^{(n)} - x_i^{(m)}|} \to 0$. Then for any coordinate i, $|x_i^{(n)} - x_i^{(m)}| \to 0$ and, hence, $\exists \lim\limits_{n\to\infty} x_i^{(n)} = x_i$. Since R^∞ is the space of all sequences, then $x = (x_n)_{n=1}^{\infty} \in R^\infty$, and R^∞ is complete.

(c) Is it possible to introduce in R^∞ a norm associated with the above metric (i.e., $d(x,y) = \|x - y\|$)?

No, it must be $\|x\| = d(x,0) = \sum_{i=1}^{\infty} 2^{-i}\frac{|x_i|}{1+|x_i|}$ but $\|\lambda x\| \ne |\lambda| \cdot \|x\|$.

11. Prove that any continuous mapping of interval $[0,1]$ into itself has at least one fixed point.

Graph the function and observe that the graph intersects the bisector $y = x$.

12. Consider operator $A: \ C[0,1] \mapsto C[0,1]$, $Ax(t) = \lambda \int\limits_0^t x(s)ds + 1$.

(a) Prove that A is contractive if $|\lambda| < 1$.

Just by definition.

(b) Find its fixed point for $\lambda = 1/2$.

$x_0(t) \equiv 0$, $x_1(t) = Ax_0 \equiv 1$, $x_2(t) = Ax_1(t) = 1 + \lambda t$, $x_3(t) = 1 + \lambda t + \frac{1}{2!}\lambda^2 t^2$, $\ldots$, $x_n = \sum_{i=0}^n \frac{\lambda^i t^i}{i!} \to e^{\lambda t}$.

(c) Does A possess a fixed point if $|\lambda| \geq 1$?

Yes, see (b). The series is convergent for any λ and t.

13. In R^N every closed bounded set is compact. Consider R^∞ with metric $d(x,y) = \sum_{i=1}^\infty \frac{1}{2^i} \frac{|x_i - y_i|}{1+|x_i - y_i|}$. Whether every closed bounded set in R^∞ is compact?

Yes, it is compact. The boundedness in R^∞ implies that $\forall i$, the i-coordinates fall into a bounded interval $[a_i, b_i]$. Let us now show that the set is totally bounded. We construct a finite ε-net as follows: choose N such that $\sum_{i=N+1}^\infty 2^{-i} = 2^{-N} < \frac{\varepsilon}{2}$ and then construct grid points for every interval $[a_i, b_i]$ for $1 \leq i \leq N$ such that $\forall i \leq N$ the distance between the neighboring grid points is less than ε/N. The number of such points for every coordinate i depends on the length of the interval $[a_i, b_i]$.

14. Whether the set $A = \{x \in l_4 : \ |x_n| \leq \frac{1}{n^{1/4}}\}$ is compact in l_4? Why?

No, since this set is unbounded in l_4.

15. Whether every bounded closed subset of a metric space is compact? If no, find a counterexample.

No. In l_∞ the set of unit vectors $x_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ is bounded, closed (why?) but not totally bounded.

16. Whether a uniformly bounded set of infinite vectors from l_∞, which satisfy the following condition

$$\forall n, m : \ |x_n - x_m| \leq C|n - m|$$

is compact in l_∞?

No. See the example from the previous problem.

17. Determine whether the following sets in $C[0,1]$ are relatively compact (pre-compact):

(a) $x_n(t) = \tan(nt)$

Not pre-compact since $x_n \notin C[0,1]$ for $n \geq 2$.

(b) $x_n(t) = \sin(nt)$

Not, since the slopes at $t = 0$ are not uniformly bounded. Then the equicontinuity condition of the Arzela theorem fails.

18. Let $x_n \to x$. Show that $\|x_n\| \to \|x\|$.

Solution. We know $\|x_n - x\| \to 0$. Then the use of the triangle inequality yiels

$$\|x_n\| - \|x\| = \|(x_n - x) + x\| - \|x\| \leq \|x_n - x\| + \|x\| - \|x\| = \|x_n - x\| \to 0.$$

Hence,

$$\forall \varepsilon > 0 \ \exists N \ \forall n > N, \ \|x_n\| - \|x\| < \varepsilon. \tag{1}$$

From another point of view,

$$\|x\| - \|x_n\| = \|(x - x_n) + x_n\| - \|x_n\| \leq \|x - x_n\| + \|x_n\| - \|x_n\| = \|x - x_n\| \to 0.$$

Hence,

$$\forall \varepsilon > 0 \ \exists N \ \forall n > N, \|x\| - \|x_n\| < \varepsilon. \tag{2}$$

From (1) and (2) we obtain

$$\Big| \|x_n\| - \|x\| \Big| < \varepsilon$$

that implies $\|x_n\| \to \|x\|$.