Stabilization and tracking via output feedback for the nonlinear benchmark system *

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Submitted to *Automatica*, Revised August 20, 1997 and Finalized January 5, 1998

Abstract

In this paper, we solve the problems of output-feedback semiglobal stabilization and tracking for the nonlinear benchmark RTAC example when the angle of the proof mass and/or the translational position of the cart are considered as the output. The proposed results complement previous work which requires exact knowledge of the translational and/or angular velocities but achieves global stabilization or global tracking. As is demonstrated in simulations, the output-feedback stabilizing and tracking controllers proposed in this paper yield performance as good as their counterparts with state feedback.

Keywords : Nonlinear control; stabilization; tracking; semiglobal; output feedback; observers.

1 Introduction

The Rotational/Translational ACtuator (RTAC) system, first introduced in (Wan *et al.* 1994), serves as a nonlinear benchmark example in a number of recent papers (Wan *et al.* 1994, Bupp *et al.* 1995, Jankovic *et al.* 1996, Kanellakopoulos and Zhao 1995, Tsiotras *et al.* 1995) for purposes of comparing and testing different control methodologies. As far as global stabilization is concerned, two main ideas have been utilized: integrator backstepping and passivity. The advantages and drawbacks of each approach have been discussed in detail in (Jankovic *et al.* 1996). Unfortunately, most globally asymptotically stabilizing controllers proposed so far require the information of a full-state or partial-state including velocities of the nonlinear RTAC system.

Motivated by the fact that requiring the precise measurement of translational and angular velocities of a mechanical system is often difficult or costs too much in practice, we consider in this paper the problem of output feedback stabilization. For the RTAC system, the angle of the proof mass and/or the translational displacement of the cart are regarded as the output. We do this at the price of achieving only semiglobal stabilization, that is, given any compact set S where the initial conditions depart from, there exists an output feedback controller such that the equilibrium of interest is asymptotically stable with a basin of attraction containing the compact set S. The interested reader is referred to (Bacciotti 1992, Khalil 1993, Teel and Praly 1995) for motivation and references for semiglobal stabilization.

More interestingly, we show that the state estimation method proposed here for the semiglobal stabilization of the RTAC system can be extended to the semiglobal tracking problem with output feedback (also see (Jiang and Hill 1997)). This is possible by using the physical structure of the benchmark RTAC system. As in the case of semiglobal stabilization, the design of semiglobal

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output-feedback tracking controllers is based on a "Separation Principle", i.e. state feedback tracking controllers and a reduced-order state observer are treated separately.

The paper is organized as follows. Section 2 contains the system description and the formulation of our problems. Section 3 is devoted to the semiglobal stabilization of the nonlinear benchmark system. Section 4 focuses on the semiglobal output feedback tracking problem. Some concluding comments are contained in Section 5.

2 RTAC model and problem statement

The picture drawn in Figure 1 represents the benchmark Rotational/Translational ACtuator (RTAC) system which is composed of a translational oscillator connected with a linear spring of stiffness k fixed at a wall and having an eccentric rotational proof.

Assume the cart moves only in the horizontal direction. Because the rotor axle is vertical, there is no gravitation effect. The motion of the oscillations is described by:

$$(M+m)\ddot{x}_c + mr(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + kx_c = F \tag{1}$$
$$(I+mr^2)\ddot{\theta} + m\ddot{x}_c r\cos\theta = N \tag{2}$$

where F is the disturbance force acting on the cart, N is the control torque applied to the proof mass, x_c is the translational position and θ is the rotational angle. In (1) and (2), M represents the total mass of the disk and the cart, m and I represent the mass and the moment of inertia of the eccentric mass, respectively, and r is the radius of rotation.

After suitable normalized transformation in the space of state variables, control input and time, the equations (1) and (2) are rewritten as (see (Wan *et al.* 1994)):

$$\ddot{x}_d + x_d = \varepsilon(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta) + F_d \tag{3}$$
$$\ddot{\theta} = u - \varepsilon\ddot{x}_d\cos\theta \tag{4}$$

where $0<\varepsilon<1$ and u is the control input.

Introducing the change of coordinates:

$$\begin{aligned} x_1 &= x_d + \varepsilon\sin\theta \\ x_2 &= \dot{x}_d + \varepsilon\dot{\theta}\cos\theta \\ x_3 &= \theta \\ x_4 &= \dot{\theta}\ , \end{aligned} \tag{5}$$

(3) and (4) are put into the following form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon\sin x_3 + F_d \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3}\Big(x_1 - \varepsilon(1+x_4^2)\sin x_3 - F_d\Big) + \frac{1}{1-\varepsilon^2\cos^2 x_3}u \end{aligned} \tag{6}$$

In the remainder of the paper, we assume that there is no disturbance, i.e. $F_d = 0$. It is further assumed that the translational and angular velocities are not available for feedback. That is to say, x_2 and x_4 are unmeasured. In previous work (Wan *et al.* 1994, Bupp *et al.* 1995, Jankovic *et al.* 1996, Kanellakopoulos and Zhao 1995), global stabilization is achieved by means of full-state feedback or partial-state feedback which requires the measurements of linear or angular velocities. In this paper, we will prove that the system (6) is asymptotically stabilizable at the equilibrium of all states zero using only the measurements of x_3 and/or x_1. The semiglobal tracking problem will also be discussed. The price to be paid in the process of this generalization is that we will assume an *a priori* bound on the initial conditions of the system state x is available for controller design. Nevertheless, this bound can be any finite positive constant.

More precisely, denoting $y := (x_1, x_3)$, our problems are formulated as follows.

Semiglobal Output Feedback Stabilization Problem. Given any compact set $\mathcal{S}_x \subset \mathrm{I\!R}^4$, which is a neighborhood of the origin, there exist a dynamic output feedback control law of the form

$$\dot{\sigma} = \nu(y, \sigma) \ , \quad u = \mu(y, \sigma) \tag{7}$$

and a compact set $\mathcal{S}_\sigma$ such that the equilibrium $(x, \sigma) = (0, 0)$ of the closed-loop system (6)-(7) is asymptotically stable, with domain of attraction containing $\mathcal{S}_x \times \mathcal{S}_\sigma$.

Semiglobal Output Feedback Tracking Problem. Given a compact set $\mathcal{M}$, for any reference signal $x_r(t)$ in $\mathcal{M}$ with bounded time derivatives $\dot{x}_r(t)$ and $\ddot{x}_r(t)$, the tracking problem is said to be *semiglobally solvable* by output feedback for system (3)-(4) if, for any compact neighborhood $\mathcal{S}_x \subset \mathrm{I\!R}^4$ of the origin, there exist a dynamic output feedback control law of the form

$$\dot{\sigma} = \nu(t, y, \sigma) \ , \quad u = \mu(t, y, \sigma) \tag{8}$$

and a compact set $\mathcal{S}_\sigma$ such that, for all initial states $(x_d(0), \dot{x}_d(0), \theta(0), \dot{\theta}(0), \sigma(0))$ in $\mathcal{S}_x \times \mathcal{S}_\sigma$, the solutions of the closed loop system (3), (4) and (8) are bounded. Furthermore, the tracking error $x_d(t) - x_r(t) \to 0$ as $t \to +\infty$.

When such an asymptotic tracking controller can be found regardless of all prescribed compact sets $\mathcal{S}_x$, the tracking problem is said to be *globally solvable* by output feedback.

The interested reader is referred to (Teel and Praly 1995) for a definition of semi-global output feedback stabilizability for general nonlinear systems. It is worth noting that the system (6) is not globally feedback linearizable (Isidori 1995) and does not satisfy the global Lipschitz condition because a quadratic nonlinearitiy appears in (6) in terms of the unmeasured state x_4. Therefore global output feedback stabilization algorithms proposed in the past literature – see (Praly and Jiang 1993) and references therein – are not applicable to system (6). See, nevertheless, Remark 1 below. Of course, for the same reasons, common global tracking methods – see, e.g., (Marino and Tomei 1995, Freeman and Kokotović 1996) – are not applicable either to system (6) to solve the above-mentioned output feedback tracking problem.

We choose to separate the semiglobal stabilization problem from the semiglobal tracking issue for the following reasons. As far as the nonlinear RTAC system is concerned, as said before, the authors of (Jankovic *et al.* 1996) gave a variety of stabilizing state-feedback controllers on the basis of backstepping and passivity concepts. In some situations, the passivity-based controllers have rather simple expressions and outperform the backstepping-based stabilizers. We demonstrate that for any GAS (globally asymptotically stablizing) and LES (locally exponentially stabilizing) state feedback control law for the RTAC system, an output feedback version can always be obtained. However, in the tracking case, we lose the flexibility of feedback designs and, in particular, it is still an open issue to find a passivity-based tracking controller for the RTAC example.

3 Semiglobal stabilization

This section is divided into two parts: we first consider the case where only the rotational angle θ is measured and considered as the output. Then we deal with the case where both the angle and the position are used for controller design.

3.1 Using only angle measurements

In this section, the output is chosen as $x_3 = \theta$. A locally asymptotically stabilizing feedback law using only angle measurements was proposed in (Kanellakopoulos and Zhao 1995). However, the stability region is small. The objective here is to achieve semi-global stabilization, i.e. with an arbitrarily large region of attraction.

As shown in (Jankovic *et al.* 1996), the following passivity-based feedback law

$$u \;=\; -k_1 x_3 \;-\; k_2 x_4 \;, \quad k_1,\; k_2 > 0 \tag{9}$$

makes the system (6) GAS, with respect to the storage function W defined by

$$W \;=\; \frac{1}{2}(x_1 - \varepsilon \sin x_3)^2 + \frac{1}{2}x_2^2 + \frac{k_1}{2}x_3^2 + \frac{1}{2}x_4^2(1 - \varepsilon^2 \cos^2 x_3) \tag{10}$$

However, (9) is a state-feedback controller and depends on the unmeasured state component x_4. The idea here is to introduce an observer to reconstruct x_4 and to prove that the observer-based output feedback controller $u = -k_1 x_3 - k_2 \widehat{x}_4$ solves the semi-global output-feedback stabilization problem stated in Section 2.

Towards this end, let $\xi = x_4 - L_1 x_3$ where L_1 is a positive constant. We have:

$$\dot{\xi} \;=\; -L_1\xi - L_1^2 x_3 - \frac{\varepsilon^2 \sin x_3 \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} + \frac{1}{1 - \varepsilon^2 \cos^2 x_3}u + \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3}\left(x_1 - \varepsilon x_4^2 \sin x_3\right) \tag{11}$$

Introduce the following observer:

$$\dot{\widehat{\xi}} \;=\; -L_1\widehat{\xi} - L_1^2 x_3 - \frac{\varepsilon^2 \sin x_3 \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} + \frac{1}{1 - \varepsilon^2 \cos^2 x_3}u \;, \quad \widehat{\xi}(0) = -L_1 x_3(0) \tag{12}$$

Setting $e = \widehat{\xi} - \xi$, we obtain:

$$\dot{e} \;=\; -L_1 e - \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3}\left(x_1 - \varepsilon x_4^2 \sin x_3\right) \;, \quad e(0) = -x_4(0) \tag{13}$$

Therefore, we establish the following result. The proof is motivated by earlier work on issues related to semi-global output feedback stabilization – see, e.g., (Teel and Praly 1995, Khalil and Esfandiari 1993), but simplification is possible using the special structure of the RTAC system.

Proposition 1 *The Semi-Global Output-Feedback Stabilization Problem is solvable for system (6) by using only θ measurements.*

Proof. We wish to prove that, for any compact set $\mathcal{S}_x \subset \mathbb{R}^4$, there exist a positive constant L_{1*} and a compact set $\mathcal{S}_{\widehat{\xi}}$ such that for all $L_1 > L_{1*}$, the closed-loop system comprised of (6), (12) and

$$u \;=\; -k_1 x_3 \;-\; k_2 \widehat{x}_4 \tag{14}$$

where $\widehat{x}_4 = \widehat{\xi} + L_1 x_3$ and $k_1 \geq k_2 > 0$, is asymptotically stable at the origin with a basin of attraction containing $\mathcal{S}_x \times \mathcal{S}_{\widehat{\xi}}$.

Noticing that $\widehat{x}_4 = x_4 + e$ and $x = (x_1, \ldots, x_4)$, in coordinates (x, e), this closed loop system is expressed as:

$$\dot{x} \;=\; f(x, e) \tag{15}$$

$$\dot{e} \;=\; -L_1 e + g(x) \tag{16}$$

where

$$f(x,e) \;=\; \begin{bmatrix} x_2 \\ -x_1 + \varepsilon \sin x_3 \\ x_4 \\ \dfrac{1}{1 - \varepsilon^2 \cos^2 x_3}(-k_1 x_3 - k_2 x_4 - k_2 e + \varepsilon x_1 \cos x_3 - \varepsilon^2(1 + x_4^2)\sin x_3 \cos x_3) \end{bmatrix} \tag{17}$$

$$g(x) \;=\; -\frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3}\left(x_1 - \varepsilon x_4^2 \sin x_3\right) \tag{18}$$

It was shown in (Jankovic *et al.* 1996) that the zero equilibrium of the system $\dot{x} = f(x,0)$ is GAS. By a Converse Lyapunov Theorem (Kurzweil 1956), there exists a positive definite and proper Lyapunov function V_0 such that

$$\frac{\partial V_0}{\partial x}(x) f(x,0) \ \leq \ -\alpha(|x|) \tag{19}$$

where α is a class K-function (see (Khalil 1996, p. 135) for a definition).

Consider the Lyapunov function candidate

$$V(x,e) \ = \ V_0(x) \ + \ \frac{1}{2}e^2 \ . \tag{20}$$

For any $\rho > 0$ and any $c > \rho$, we prove in the sequel that there exists an $L_{1*} > 0$ such that for all $L_1 > L_{1*}$ the time derivative of V along solutions of the closed loop system is negative for all initial conditions starting from the set:

$$\mathcal{S}_{xe} \ = \ \{(x,e) \ : \ \rho \leq V(x,e) \leq c\} \tag{21}$$

Differentiating V along the solutions of the system (6), (12) and (14) yields:

$$\dot{V} \ \leq \ -\alpha(|x|) + \frac{\partial V_0}{\partial x}(x) \left(f(x,e) - f(x,0)\right) - L_1 e^2 + e g(x) \tag{22}$$

Since $\mathcal{S}_{xe}$ is a compact set, there exists a positive constant k_3 such that, for all (x,e) in $\mathcal{S}_{xe}$,

$$\left|\frac{\partial V_0}{\partial x}(x)(f(x,e) - f(x,0))\right| \leq k_3|e| \ , \quad |g(x)| \leq k_3 \tag{23}$$

Thus, (22) implies:

$$\dot{V} \ \leq \ -\alpha(|x|) - L_1 e^2 + 2k_3|e| \tag{24}$$

From (24), by applying Lemma 2.1 of (Teel and Praly 1995), or by contradiction, there exists a positive constant L_{1*} such that for all $L_1 > L_{1*}$, we have:

$$\dot{V} \ < \ 0 \quad \forall (x,e) \in \mathcal{S}_{xe} \tag{25}$$

From (25) and together with the fact that the zero equilibrium of $\dot{x} = f(x,0)$ is LES (locally exponentially stable), we can invoke (Teel and Praly 1995, Prop. 5.1) to conclude Proposition 1. However, in the present case, there is a simpler argument. Using the LES property of $\dot{x} = f(x,0)$ at $x = 0$, it is not difficult to prove that, for sufficiently large L_1, there is a neighborhood $\mathcal{N}$ of the origin $(x,e) = (0,0)$ which is independent of L_1 and is contained in the basin of attraction of the closed-loop system (15) and (16). Since $\rho > 0$ is arbitrary and can be chosen so small that $\{(x,e) : V \leq 2\rho\}$ is contained in $\mathcal{N}$, it follows from (25) that all the solutions starting from $\mathcal{S}_{xe}$ under the constraint $e(0) = -x_4(0)$ are ultimately bounded and will eventually enter into the set $\mathcal{N}$. □

Remark 1 Since the submission of our manuscript, we became aware of the work (Escobar and Ortega 1997) where a GAS control law without velocity measurements is proposed on the basis of energy shaping and damping injection. An anonymous reviewer did point out that, from (9) and using passive systems theory, we can also deduce a GAS dynamic output feedback law defined by

$$\dot{\chi} \ = \ -\frac{1}{\tau_3}\chi - \frac{1}{\tau_3^2}x_3 \tag{26}$$

$$u \ = \ -\tau_1 x_3 - \tau_2 \chi - \frac{\tau_2}{\tau_3}x_3 \tag{27}$$

where τ_1, τ_2 and τ_3 are any positive constants.

3.2 Using angle and position measurements

As pointed out in (Jankovic *et al.* 1996), in order to ensure a quick settling for the translational position, we have to use x_1 in the feedback law. The following state-feedback controller has been proposed in (Jankovic *et al.* 1996), instead of (9),

$$u = -k_0\varepsilon\cos x_3(-x_1+\varepsilon\sin x_3) - k_1x_3 - k_2x_4 \tag{28}$$

which globally asymptotically stabilizes the RTAC system with respect to the storage function:

$$W_a = \frac{k_0+1}{2}[(x_1-\varepsilon\sin x_3)^2+x_2^2] + \frac{k_1}{2}x_3^2 + \frac{1}{2}x_4^2(1-\varepsilon^2\cos^2 x_3) \tag{29}$$

Analogously, using the observer (12) and setting $\widehat{x}_4 = \widehat{\xi} + L_1x_3$, the output feedback law derived from (28) with x_4 replaced by $\widehat{x}_4$ is a semiglobally stabilizing controller. The proof is similar to the proof of Proposition 1. As the simulations show (see Figures 2 and 3), the output-feedback controller using output (x_1, x_3)

$$u = -k_0\varepsilon\cos x_3(-x_1+\varepsilon\sin x_3) - k_1x_3 - k_2\widehat{x}_4 \tag{30}$$

results in a smaller settling time for the translational motion x_1 than the previous one (14) using only x_3 measurements.

In fact, when $y = (x_1, x_3)$ is considered as output, we can prove the following general result for the RTAC system.

Proposition 2 *For any GAS and LES state-feedback controller $\overline{u}(x)$ for system (6), there exists always an observer-based output feedback controller $\overline{u}(x_1, \widehat{x}_2, x_3, \widehat{x}_4)$, with*

$$\dot{\widehat{\zeta}} = \nu_1(\widehat{\zeta}, y)\ , \quad \widehat{x}_2 = \widehat{\zeta} + L_1x_1 \tag{31}$$

$$\dot{\widehat{\xi}} = \nu_2(\widehat{\xi}, y)\ , \quad \widehat{x}_4 = \widehat{\xi} + L_1x_3 \tag{32}$$

which solves the Semi-Global Output-Feedback Stabilization Problem for system (6).

Proof. In addition to the above observer (12) for the reconstruction of the unmeasured state x_4, we need to build an observer for the unmeasured state x_2. Letting $\zeta = x_2 - L_1x_1$, we have:

$$\dot{\zeta} = -L_1\zeta - L_1^2x_1 - x_1 + \varepsilon\sin x_3 \tag{33}$$

Introducing an observer as:

$$\dot{\widehat{\zeta}} = -L_1\widehat{\zeta} - L_1^2x_1 - x_1 + \varepsilon\sin x_3\ , \quad \widehat{\zeta}(0) = -L_1x_1(0) \tag{34}$$

and denoting $e_2 = \widehat{\zeta} - \zeta$, it holds:

$$\dot{e}_2 = -L_1e_2\ , \quad e_2(0) = -x_2(0) \tag{35}$$

Letting $\overline{e} = (e, e_2)^T$, the closed loop system (6) with $u = \overline{u}(x_1, \widehat{x}_2, x_3, \widehat{x}_4)$, where $\widehat{x}_2 = \widehat{\zeta} + L_1x_1$ and $\widehat{x}_4 = \widehat{\xi} + L_1x_3$ as defined in the subsection 3.1, is rewritten as:

$$\dot{x} = \overline{f}(x, \overline{e}) \tag{36}$$

$$\dot{\overline{e}} = -L_1\overline{e} + \overline{g}(x) \tag{37}$$

where

$$\overline{f}(x,\overline{e}) = \begin{bmatrix} x_2 \\ -x_1+\varepsilon\sin x_3 \\ x_4 \\ \kappa(x) + \dfrac{1}{1-\varepsilon^2\cos^2 x_3}\overline{u}(x_1, x_2+e_2, x_3, x_4+e) \end{bmatrix} \tag{38}$$

$$\overline{g}(x) = \begin{bmatrix} -\dfrac{\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3}\left(x_1 - \varepsilon x_4^2\sin x_3\right) \\ 0 \end{bmatrix} \tag{39}$$

where $\kappa(x) = \frac{\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3}(x_1 - \varepsilon(1+x_4^2)\sin x_3)$. The proof for the semi-global stabilizability of (36) and (37) follows the same lines as the proof of Proposition 1. □

One may question the practical usefulness of the output-feedback controllers (14) and (30) since the observer gain L_1 depends on the Lyapunov function V_0 of the closed-loop system with state-feedback which may be hard to construct. However, the output-feedback controllers (14) and (30) are explicit functions of gain L_1. Desired performance for the RTAC system in practice can be achieved by increasing the design parameter L_1 until responses are satisfactory, as illustrated in the following simulations. It can be noted that this value can be chosen to work for a given set of initial conditions.

3.3 Simulation results

Figure 2 gives the simulation for the normalized position x_1 and the control torque u with the output feedback control law (14) whereas Figure 3 gives corresponding results with the output feedback control law (30). In these simulations we chose the following values for initial conditions and design parameters:

$$x(0) = (1,1,1,1)\ ,\quad e(0) = -1\ ,\quad L_1 = 10\ ,\quad k_0 = 10\ ,\quad k_1 = 1\ ,\quad k_2 = 0.5\ ,\quad \varepsilon = 0.1 \qquad (40)$$

We observe that, as in (Jankovic *et al.* 1996) with state-feedback, the output feedback control law (30) yields faster convergence for x_1 than the output feedback control law (14) based only on the measurement of θ. Also, according to our simulations, we observe that the output-feedback control laws proposed in this paper ensure performance as good as the passivity-based state-feedback controllers in (Jankovic *et al.* 1996) when the design parameter L_1 is chosen sufficiently large.

4 Tracking

The purpose of this section is to extend the semiglobal stabilization scheme to the semiglobal tracking case. That is, we want the dimensionless cart displacement x_d in (3)-(4) to track semiglobally a reference trajectory x_r with output feedback when $y = (x_d, \theta)$ is seen as the output. The corresponding problem with state feedback was considered in (Kanellakopoulos and Zhao 1995) using the idea of internal models.

The control design procedure suggested in (Kanellakopoulos and Zhao 1995) is incomplete; a final statement of a state-feedback controller and stability result are not given. As the development here of output-feedback asymptotic tracking is based on the state-feedback case, we first formulate a complete backstepping design of state-feedback tracking controllers for the system (3)-(4). In particular, the class of reference signals to be tracked is clearly identified and the stronger property of global *uniform* asymptotic stability is obtained for the closed-loop differential equations after the tracking state feedback is applied.

4.1 Global tracking via state feedback

As already mentioned, the disturbance-free system (3)-(4) is not feedback linearizable. A careful analysis implies that usual global tracking algorithms with state-feedback (see, e.g., (Khalil 1993, Marino and Tomei 1995)) cannot be applied to system (3)-(4). However, as shown in (Kanellakopoulos and Zhao 1995), the idea of using the internal model of the reference signal plays a central role in solving the state-feedback global tracking problem.

More specifically, noticing that the dynamics x_d satisfy

$$\frac{d^2}{dt^2}\sin\theta \;=\; -\frac{1}{\varepsilon}(\ddot{x}_d \;+\; x_d)\ , \qquad (41)$$

we introduce an extra differential equation:

$$\ddot{\eta} \;=\; -\frac{1}{\varepsilon}(\ddot{x}_r + x_r) \tag{42}$$

We impose the following assumption on the reference signal x_r.

Assumption 1 The reference signal $x_r(t)$ is C^2 and periodic. Further, the solution $\eta(t)$ of (42), with appropriate initial values $(\eta(0), \dot{\eta}(0))$, is periodic and is overbounded by a constant $\eta_{\max}$ in $(0, 1)$.

Simple examples of such reference signals $x_r(t)$ include $x_r \equiv 0$ [in this case, $\dot{\eta}(0) = 0$ and $\eta(0) \in (-1, 1)$] and $x_r(t) = r_0 \sin(\omega_0 t)$ with $\omega_0 \neq 0$ and $|r_0(1-\omega_0^{-2})| < \varepsilon$ (in this case, $\dot{\eta}(0) = \varepsilon^{-1} r_0(\omega_0^{-1} - \omega_0)$ and $|\eta(0)| < 1 - \varepsilon^{-1}|r_0(1-\omega_0^{-2})|$).

As in (Kanellakopoulos and Zhao 1995), we introduce the following reference signals for θ and $\dot{\theta}$:

$$\theta_r = \arcsin(\eta) \;, \quad \dot{\theta}_r = \frac{1}{\cos\theta_r}\dot{\eta} \tag{43}$$

Then, we consider the following new variables, instead of (5),

$$\begin{aligned} z_1 &= x_d + \varepsilon \sin\theta - x_r - \varepsilon\eta \\ z_2 &= \dot{x}_d + \varepsilon\dot{\theta}\cos\theta - \dot{x}_r - \varepsilon\dot{\eta} \\ z_3 &= \theta \;-\; \theta_r \\ z_4 &= \dot{\theta} \;-\; \dot{\theta}_r \end{aligned} \tag{44}$$

It is directly checked that, under the above z-coordinates, the system (3)-(4) with $F_d = 0$ is transformed into:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 \;+\; \varepsilon(\sin(z_3 + \theta_r) - \sin\theta_r) \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= \frac{\varepsilon\cos\theta}{1 - \varepsilon^2\cos^2\theta}\left(x_d + \varepsilon\sin\theta - \varepsilon(1 + \dot{\theta}_r^2 + 2\dot{\theta}_r z_4 + z_4^2)\sin\theta\right) \\ &\quad + \frac{1}{1 - \varepsilon^2\cos^2\theta}u \;-\; \frac{\ddot{\eta}}{\cos\theta_r} \;-\; \frac{\eta\dot{\eta}^2}{\cos^3\theta_r} \end{aligned} \tag{45}$$

Note that θ_r, $\dot{\theta}_r$, η, $\dot{\eta}$ and $\ddot{\eta}$ are signals available for feedback.

It is important to note that, thanks to Assumption 1,

$$|\eta(t)| \;\leq \eta_{\max} \;<\; 1 \,, \quad |\theta_r(t)| \;\leq \theta_{\max} \;<\; \frac{\pi}{2} \,, \quad \forall\, t \geq 0 \tag{46}$$

With this observation, we can employ the backstepping method to design a globally stabilizing state feedback controller for system (45) which, in turn, globally solves the tracking problem via state feedback for the original system (3)-(4).

To start the backstepping design procedure, we consider the quadratic function $V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$ for the (z_1, z_2)-subsystem of (45). Following the calculations in (Kanellakopoulos and Zhao 1995) and denoting

$$\tilde{z}_3 \;=\; z_3 \;+\; l_1 \arctan z_2 \tag{47}$$

where $0 < l_1 < 2(1 - 2\theta_{\max}/\pi)$, differentiating V_1 along solutions of system (45) gives:

$$\begin{aligned} \dot{V}_1 &= \varepsilon z_2\left[\sin(z_3 + \theta_r) - \sin(-l_1\arctan z_2 + \theta_r)\right] + \varepsilon z_2\left[\sin(-l_1\arctan z_2 + \theta_r) - \sin\theta_r\right] \\ &= 2\varepsilon z_2 \sin\frac{\tilde{z}_3}{2}\cos\frac{z_3 - l_1\arctan z_2 + 2\theta_r}{2} \;-\; 2\varepsilon z_2\sin\frac{l_1\arctan z_2}{2}\cos\frac{l_1\arctan z_2 - 2\theta_r}{2} \end{aligned} \tag{48}$$

By virtue of (46) and the choice of design parameter l_1, we have

$$\left|\frac{l_1\arctan z_2 - 2\theta_r}{2}\right| \;\leq\; \frac{\pi}{4}l_1 \;+\; \theta_{\max} \;<\; \frac{\pi}{2} \tag{49}$$

Then,

$$z_2 \sin \frac{l_1 \arctan z_2}{2} \cos \frac{l_1 \arctan z_2 - 2\theta_r}{2} \ \geq \ 0 \tag{50}$$

where the equality holds at some time instant $t \geq 0$ if and only if $z_2(t) = 0$.

Next, consider the candidate Lyapunov function $V_2 = V_1 + \frac{1}{2}\widetilde{z}_3^2$ for the (z_1, z_2, z_3)-subsystem of (45) with z_4 viewed as virtual control. With (48), the time derivative of V_2 satisfies:

$$\begin{aligned} \dot{V}_2 \ &= \ 2\varepsilon z_2 \sin \frac{\widetilde{z}_3}{2} \cos \frac{z_3 - l_1 \arctan z_2 + 2\theta_r}{2} \ - \ 2\varepsilon z_2 \sin \frac{l_1 \arctan z_2}{2} \cos \frac{l_1 \arctan z_2 - 2\theta_r}{2} \\ & \quad + \widetilde{z}_3 \left[z_4 + l_1 \sec^2(\arctan z_2) \left(-z_1 + \varepsilon \sin(z_3 + \theta_r) - \varepsilon \sin \theta_r \right) \right] \end{aligned} \tag{51}$$

Noticing that

$$2 \sin \frac{\widetilde{z}_3}{2} \ = \ \widetilde{z}_3 \int_0^1 \cos(\frac{\widetilde{z}_3}{2}\lambda) d\lambda \tag{52}$$

and denoting

$$\begin{aligned} \beta_1(\theta_r, z_1, z_2, z_3) \ &= \ -\varepsilon z_2 \cos \frac{z_3 - l_1 \arctan z_2 + 2\theta_r}{2} \int_0^1 \cos(\frac{\widetilde{z}_3}{2}\lambda) d\lambda \\ & \quad - l_1 \sec^2(\arctan z_2) \left(-z_1 + \varepsilon \sin(z_3 + \theta_r) - \varepsilon \sin \theta_r \right) \ - \ l_2 \widetilde{z}_3 \end{aligned} \tag{53}$$

with $l_2 > 0$, when substituting $z_4 := \widetilde{z}_4 + \beta_1$ into (51), we get:

$$\dot{V}_2 \ = \ -2\varepsilon z_2 \sin \frac{l_1 \arctan z_2}{2} \cos \frac{l_1 \arctan z_2 - 2\theta_r}{2} \ - \ l_2 \widetilde{z}_3^2 \ + \ \widetilde{z}_3 \widetilde{z}_4 \tag{54}$$

Finally, we consider the following candidate Lyapunov function

$$V_3 \ = \ V_2 \ + \ \frac{1}{2}\widetilde{z}_4^2 \ = \ \frac{1}{2}z_1^2 \ + \ \frac{1}{2}z_2^2 \ + \ \frac{1}{2}(z_3 + l_1 \arctan z_2)^2 \ + \ \frac{1}{2}(z_4 - \beta_1)^2 \tag{55}$$

for the whole system (45).

Taking the time differentiation of V_3 with respect to the solutions of (45), we have:

$$\begin{aligned} \dot{V}_3 \ &= \ -2\varepsilon z_2 \sin \frac{l_1 \arctan z_2}{2} \cos \frac{l_1 \arctan z_2 - 2\theta_r}{2} \ - \ l_2 \widetilde{z}_3^2 \ + \ \widetilde{z}_3 \widetilde{z}_4 \\ & \quad + \widetilde{z}_4 \left(\frac{1}{1 - \varepsilon^2 \cos^2 \theta} u \ + \ \Delta_1 \ - \ \dot{\beta}_1 \right) \end{aligned} \tag{56}$$

where

$$\Delta_1 \ = \ \frac{\varepsilon(x_d + \varepsilon \sin \theta) \cos \theta - \varepsilon^2 (1 + \dot{\theta}_r^2 + 2\dot{\theta}_r z_4 + z_4^2) \sin \theta \cos \theta}{1 - \varepsilon^2 \cos^2 \theta} \ - \ \frac{\ddot{\eta}}{\cos \theta_r} \ - \ \frac{\eta \dot{\eta}^2}{\cos^3 \theta_r} \tag{57}$$

$$\dot{\beta}_1 \ = \ \frac{\partial \beta_1}{\partial \theta_r} \dot{\theta}_r \ + \ \frac{\partial \beta_1}{\partial z_1} z_2 \ + \ \frac{\partial \beta_1}{\partial z_2} (-z_1 + \varepsilon \sin(z_3 + \theta_r) - \varepsilon \sin \theta_r) \ + \ \frac{\partial \beta_1}{\partial z_3} z_4 \tag{58}$$

Consequently, choosing the following control law for u:

$$u \ = \ \vartheta(x_r, \eta, \dot{\eta}, \ddot{\eta}, z_1, z_2, z_3, z_4) \tag{59}$$

$$\ \ = \ (1 - \varepsilon^2 \cos^2 \theta) \left(-l_3 \widetilde{z}_4 - \widetilde{z}_3 - \Delta_1 + \dot{\beta}_1 \right) \tag{60}$$

with $l_3 > 0$, from (56), it follows

$$\dot{V}_3 \ = \ -2\varepsilon z_2 \sin \frac{l_1 \arctan z_2}{2} \cos \frac{l_1 \arctan z_2 - 2\theta_r}{2} \ - \ l_2 \widetilde{z}_3^2 \ - \ l_3 \widetilde{z}_4^2 \tag{61}$$

Finally, we establish the following result.

Proposition 3 *Consider the class of reference signals $x_r(t)$ satisfying Assumption 1. The tracking problem is globally solvable for the system (3)-(4) by state feedback. In particular, the equilibrium $z = 0$ of the resulting closed-loop system (45) and (60) is globally uniformly asymptotically stable.*

Proof. It suffices to prove the second statement since the boundedness of $z(t)$ implies the boundedness of $x(t)$ and the convergence of $z_1(t)$ and $z_3(t)$ implies the convergence of the tracking error $x_d(t) - x_r(t)$.

By definition of V_3 in (55), since $\beta_1(\cdot, 0, 0, 0) \equiv 0$, the function V_3 is positive definite, decrescent and radially unbounded. From (61), the uniform stability of the equilibrium $z = 0$ follows (see, e.g., (Marino and Tomei 1995, Theorem B.1.5)). In addition, the signals $z(t)$ and $\dot{z}(t)$ are bounded for all $t \in [0, \infty)$.

As a direct application of Barbălat's lemma (see, e.g., (Khalil 1996, Lemma 4.2)), with (61) and (50), it follows that $z_2(t)$, $\tilde{z}_3(t)$ and $\tilde{z}_4(t)$ tend to zero as $t \to \infty$. As a result, $z_3(t)$ tends to zero. Then, for the z_2-equation in (45):

$$\dot{z}_2 \;=\; -z_1 \;+\; \varepsilon(\sin(z_3 + \theta_r) - \sin\theta_r) \tag{62}$$

invoking Lemma 2 of (Jiang and Nijmeijer 1997), we prove that $z_1(t)$ converges to zero as $t \to \infty$. Using the definition of $\tilde{z}_4$, we conclude that $z_4(t)$ goes to zero as $t \to \infty$. Thanks to Assumption 1, the closed-loop system (45) and (60) is periodic. Therefore, the proof of Proposition 3 is completed by means of a classical criterion on the global uniform asymptotic stability (GUAS) – see, e.g., (Hahn 1967, Theorem 38.5), (Khalil 1996, Lemma 3.3). □

Remark 2 If Assumption 1 is weakened to the condition that $x_r(t)$, $\dot{x}_r(t)$, $\ddot{x}_r(t)$ and $\dot{\eta}(t)$ are bounded and (46) holds, then we only obtain global asymptotic stability, instead of the GUAS property which is needed in Section 4.2.

In preparation for the tracking control design via output feedback in next section, we give the following corollary.

Corollary 1 *Under the conditions of Proposition 3, the zero equilibrium $z = 0$ of the closed-loop system (45) and (60) is exponentially stable.*

Proof. In light of Assumption 1 and (Khalil 1996, Theorem 3.13), Corollary 1 follows if the Taylor linearization of the nonlinear time-varying system (45) and (60)

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -z_1 \;+\; \varepsilon(\cos\theta_r) z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= (\varepsilon\cos\theta_r + l_1 l_2 + l_1 l_3) z_1 + (\varepsilon\dot{\theta}_r \sin\theta_r - l_3\varepsilon\cos\theta_r - l_1 l_2 l_3) z_2 \\
&\quad + \Big[-1 - l_1 l_3 \varepsilon\cos\theta_r - l_2 l_3 + l_1\varepsilon\dot{\theta}_r \sin\theta_r - (\varepsilon\cos\theta_r + l_1 l_2)\varepsilon\cos\theta_r\Big] z_3 - (l_1\varepsilon\cos\theta_r + l_2 + l_3) z_4
\end{aligned} \tag{63}$$

is exponentially stable at $z = 0$.

To prove that the linear time-varying system (63) is exponential stable at $z = 0$, consider the quadratic function

$$V_l \;=\; \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}(z_3 + l_1 z_2)^2 + \frac{1}{2}\left[z_4 - l_1 z_1 + (\varepsilon\cos\theta_r + l_1 l_2) z_2 + (l_1\varepsilon\cos\theta_r + l_2) z_3\right]^2 \tag{64}$$

As it can be directly checked, the time derivative of V_l along solutions of (63) satisfies:

$$\dot{V}_l \;=\; -l_1 z_2^2 \varepsilon\cos\theta_r - l_2(z_3 + l_1 z_2)^2 - l_3\left[z_4 - l_1 z_1 + (\varepsilon\cos\theta_r + l_1 l_2) z_2 + (l_1\varepsilon\cos\theta_r + l_2) z_3\right]^2 \tag{65}$$

Since V_l is positive definite, decrescent and radially unbounded, from (65), it follows that $z = 0$ is a uniformly stable equilibrium position. Further, a direct application of Barbălat's lemma yields that

$z_2(t)$, $z_3(t)$ and $z_4(t) - l_1 z_1(t)$ tend to zero as $t \to +\infty$. As in the proof of Proposition 3, applying again (Jiang and Nijmeijer 1997, Lemma 2) to the z_2-subsystem of (63), it follows that $z_1(t)$ converges to zero as $t \to +\infty$. Therefore, $z_4(t)$ converges to zero. Using Assumption 1 together with (Hahn 1967, Theorem 38.5), we conclude that $z = 0$ is a uniformly asymptotically stable solution for the linearized system (63). □

4.2 Semiglobal tracking via output feedback

Since the linear and angular velocities (i.e. $\dot{x}_d$, $\dot{\theta}$) are assumed to be unmeasured in this paper, the state variables z_2 and z_4 are not available for controller design. So the tracking control law u as defined in (60) cannot be implemented in practice.

Following the strategy used in Section 3, the goal of this section is to propose a high-gain observer to reconstruct z_2 and z_4 and substitute their estimates into the state feedback controller (60). In doing this, we sacrifice the global tracking solution but achieve semiglobal tracking.

To build a reduced-order high-gain observer for z_2 and z_4, introduce the new variables w_1 and w_2 by

$$w_1 = z_2 - L_2 z_1 , \quad w_2 = z_4 - L_2 z_3 \tag{66}$$

which satisfy

$$\dot{w}_1 = -L_2 w_1 - L_2^2 z_1 - z_1 + \varepsilon\left(\sin(z_3 + \theta_r) - \sin\theta_r\right) \tag{67}$$

$$\begin{aligned} \dot{w}_2 &= -L_2 w_2 - L_2^2 z_3 + \frac{1}{1-\varepsilon^2\cos^2\theta} u - \frac{\ddot{\eta}}{\cos\theta_r} - \frac{\eta\dot{\eta}^2}{\cos^3\theta_r} \\ &\quad + \frac{\varepsilon\cos\theta}{1-\varepsilon^2\cos^2\theta}\left(x_d + \varepsilon\sin\theta - \varepsilon(1 + \dot{\theta}_r^2 + 2\dot{\theta}_r z_4 + z_4^2)\sin\theta\right) \end{aligned} \tag{68}$$

where $L_2 > 0$ is a design parameter and (45) was used.

Then the observer is designed as follows:

$$\dot{\widehat{w}}_1 = -L_2 \widehat{w}_1 - L_2^2 z_1 - z_1 + \varepsilon\left(\sin(z_3 + \theta_r) - \sin\theta_r\right) \tag{69}$$

$$\begin{aligned} \dot{\widehat{w}}_2 &= -L_2 \widehat{w}_2 - L_2^2 z_3 + \frac{1}{1-\varepsilon^2\cos^2\theta} u - \frac{\ddot{\eta}}{\cos\theta_r} - \frac{\eta\dot{\eta}^2}{\cos^3\theta_r} \\ &\quad + \frac{\varepsilon\cos\theta}{1-\varepsilon^2\cos^2\theta}\left(x_d + \varepsilon\sin\theta - \varepsilon(1 + \dot{\theta}_r^2)\sin\theta\right) \end{aligned} \tag{70}$$

with $\widehat{w}_1(0) = -L_2 z_1(0)$ and $\widehat{w}_2(0) = -L_2 z_3(0)$. Letting

$$e_3 = \widehat{w}_1 - w_1 , \quad e_4 = \widehat{w}_2 - w_2 \tag{71}$$

we obtain:

$$\dot{e}_3 = -L_2 e_3 \tag{72}$$

$$\dot{e}_4 = -L_2 e_4 + \frac{\varepsilon^2}{1-\varepsilon^2\cos^2\theta}\left(2\dot{\theta}_r z_4 + z_4^2\right)\sin\theta\cos\theta \tag{73}$$

Thus, we can prove the following.

Proposition 4 *Consider the class of reference signals $x_r(t)$ satisfying Assumption 1. The tracking problem is semiglobally solvable for the system (3)-(4) by dynamic output feedback controller*

$$u = \vartheta(x_r, \eta, \dot{\eta}, \ddot{\eta}, z_1, \widehat{z}_2, z_3, \widehat{z}_4) \tag{74}$$

where ϑ is defined as in (59) and $\widehat{z}_2$ and $\widehat{z}_4$ are given by

$$\widehat{z}_2 = \widehat{w}_1 + L_2 z_1 , \quad \widehat{z}_4 = \widehat{w}_2 + L_2 z_3 \tag{75}$$

Proof. First note that, for each $t \geq 0$, the mapping Φ defined by (44), i.e.

$$\Phi(t, x_d - x_r, \dot{x}_d - \dot{x}_r, \theta - \theta_r, \dot{\theta} - \dot{\theta}_r) \;=\; z$$

is a global diffeomorphism from $\mathbb{R}^4$ onto $\mathbb{R}^4$ and preserves the origin. With this fact in hand, we need only to prove the following claim:

Claim. For any compact set $S_z \subset \mathbb{R}^4$, there exist a compact set $S_w \subset \mathbb{R}^2$ and a positive constant L_{2*} such that, for all $L > L_{2*}$, the equilibrium $(z, \widehat{w}_1, \widehat{w}_2) = (0,0,0)$ of (44), (69), (70) and (74) is asymptotically stable with basin of attraction containing $S_z \times S_w$.

In coordinates (z, e_3, e_4), the closed-loop system (44), (69), (70) and (74) can be rewritten in compact form:

$$\begin{aligned} \dot{z} &= f_z(t, z, e_3, e_4) \\ \dot{e}_3 &= -L_2 e_3 \\ \dot{e}_4 &= -L_2 e_4 \;+\; \frac{\varepsilon^2}{1 - \varepsilon^2 \cos^2\theta} \left(2\dot{\theta}_r z_4 + z_4^2\right) \sin\theta \cos\theta \end{aligned} \tag{76}$$

As shown in Proposition 3, the equilibrium $z = 0$ of $\dot{z} = f_z(t, z, 0, 0)$ is globally uniformly asymptotically stable. Thus, by a Lyapunov Converse Theorem (see (Hahn 1967, Sec. 49) for example), there exists a C^1 function $U_0 : \mathbb{R}_+ \times \mathbb{R}^4 \to \mathbb{R}$ that satisfies the following properties:

$$\begin{aligned} \psi_1(|z|) \;\leq\; U_0(t,z) \;&\leq\; \psi_2(|z|) \\ \frac{\partial U_0}{\partial t} \;+\; \frac{\partial U_0}{\partial z} f_z(t, z, 0, 0) \;&\leq\; -\psi_3(|z|) \\ \left|\frac{\partial U_0}{\partial z}\right| \;&\leq\; \psi_4(|z|) \end{aligned} \tag{77}$$

where ψ_i, $1 \leq i \leq 4$, are class K-functions.

Consider the Lyapunov function candidate U defined by

$$U \;=\; U_0(t,z) \;+\; \frac{1}{2} e_3^2 \;+\; \frac{1}{2} e_4^2 \tag{78}$$

As in the proof of Proposition 1, we can prove that there exists a positive real number L_{2*} so that, for all $L_2 > L_{2*}$, the time derivative of U along the solutions of (76) satisfies

$$\dot{U} \;<\; 0 \quad \forall\,(z, e_3, e_4) \in \mathcal{S}_{ze} := \{(z, e_3, e_4) \;:\; \rho \leq U(0, z, e_3, e_4) \leq c\} \tag{79}$$

where $c > \rho$ is arbitrary.

Notice that the zero equilibrium of system $\dot{z} = f_z(t, z, 0, 0)$ in (76) is exponentially stable (cf. Corollary 1). Finally, we conclude Proposition 4 by imitating the proof of Proposition 1. $\square$

4.3 Simulation results

The simulations drawn in Figures 4 and 5 were performed in MATLAB with the following choice of initial conditions and design parameters:

$$(z(0), e_3(0), e_4(0)) = (1, 0.2, 0, 0.2, -0.2, -0.2)\;, \quad L_2 = 10\;, \quad l_1 = 0.5\;, \quad l_2 = l_3 = 1\;, \quad \varepsilon = 0.1$$

The reference signals are $x_r(t) = \sin t$ and $x_r(t) = 0.1 \sin 2t$, respectively.

As our simulations show, the output feedback tracking controllers (74) guarantee performace as good as the backstepping-based state feedback tracking controllers proposed in (Kanellakopoulos and Zhao 1995) when the design parameter L_2 is selected sufficiently large.

5 Conclusion

We have revisited in this paper the stabilization and tracking problems for the nonlinear benchmark RTAC example. A semiglobal output feedback stabilization approach and a semiglobal output feedback tracking strategy are proposed using the measurements of the angle-only or both the angle and the position of the RTAC system. This complements the previous work in this topic with full-state feedback or partial-state feedback but using the measurements of linear or angular velocities. The significance of the results and the effectiveness of the proposed output-feedback controllers are demonstrated through a number of simulations.

Acknowledgement. This work was supported by the ARC Large Grant Reference No. A49530078. Z.-P. Jiang is thankful to Andrew Teel and Laurent Praly for helpful discussions. We are grateful to two anonymous reviewers for their constructive comments.

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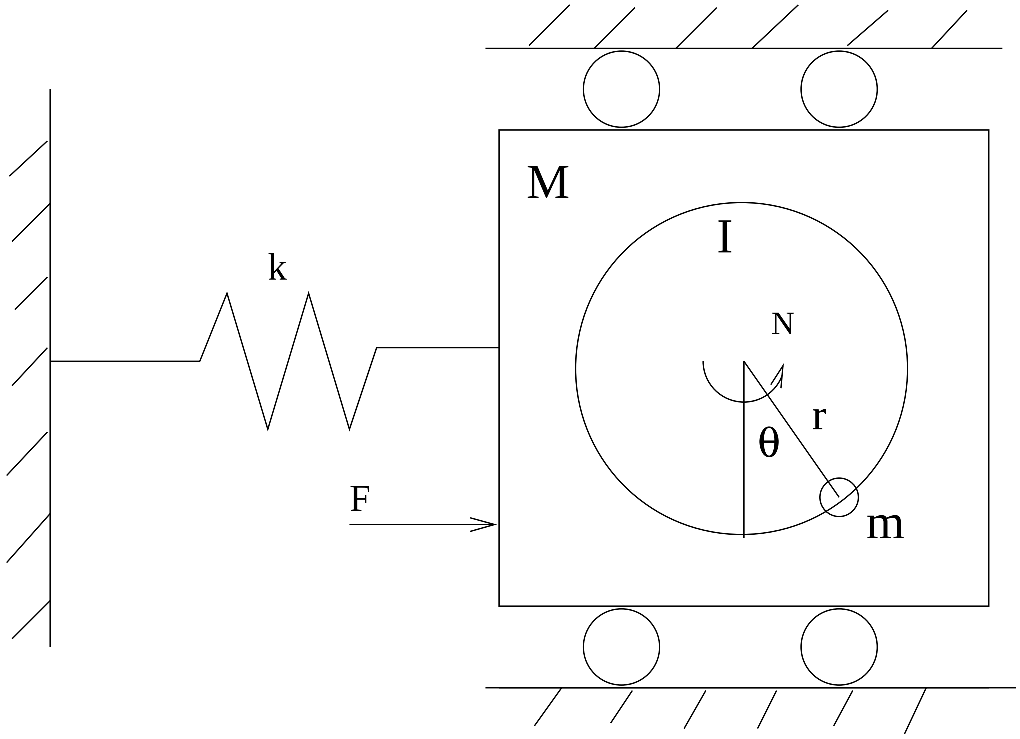

Figure 1: Nonlinear benchmark example (Wan *et al.* 1994)

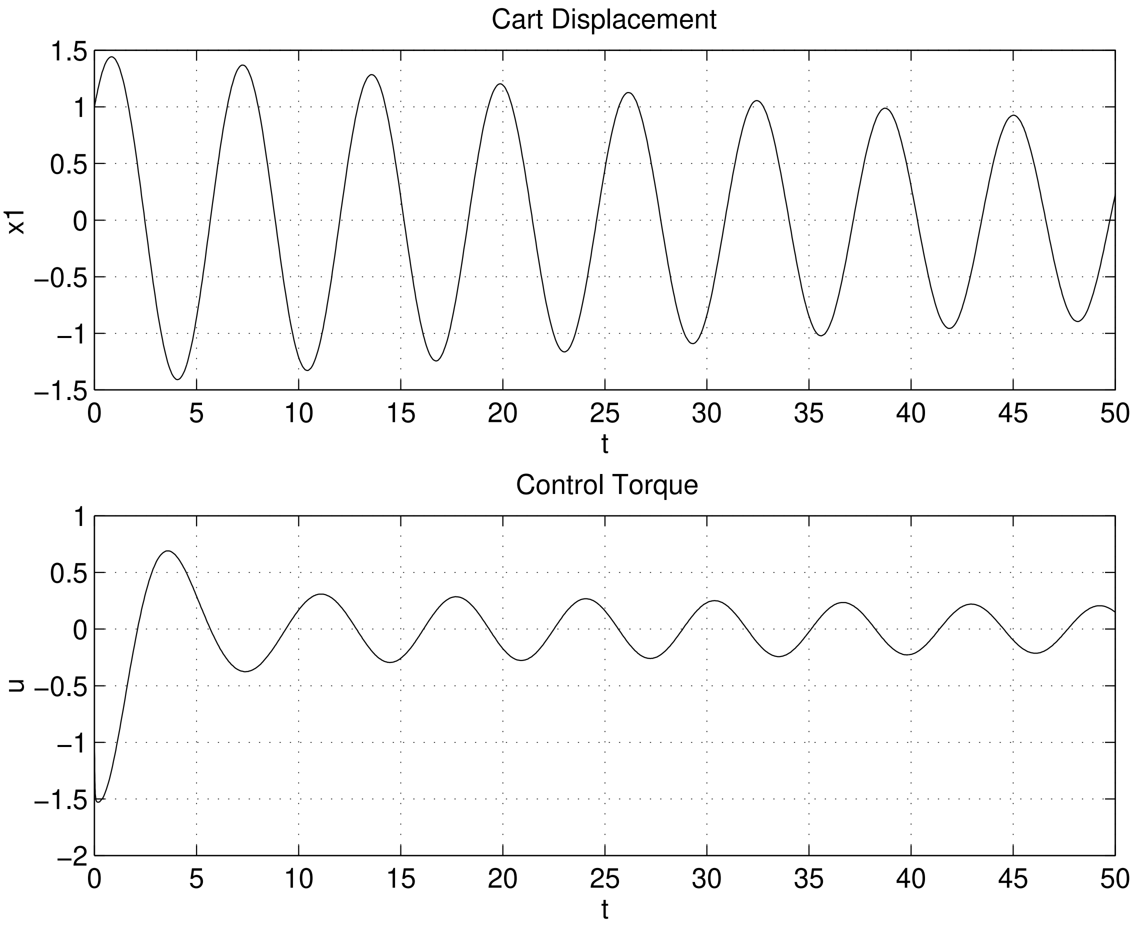

Figure 2: Stabilization with output feedback controller (14).

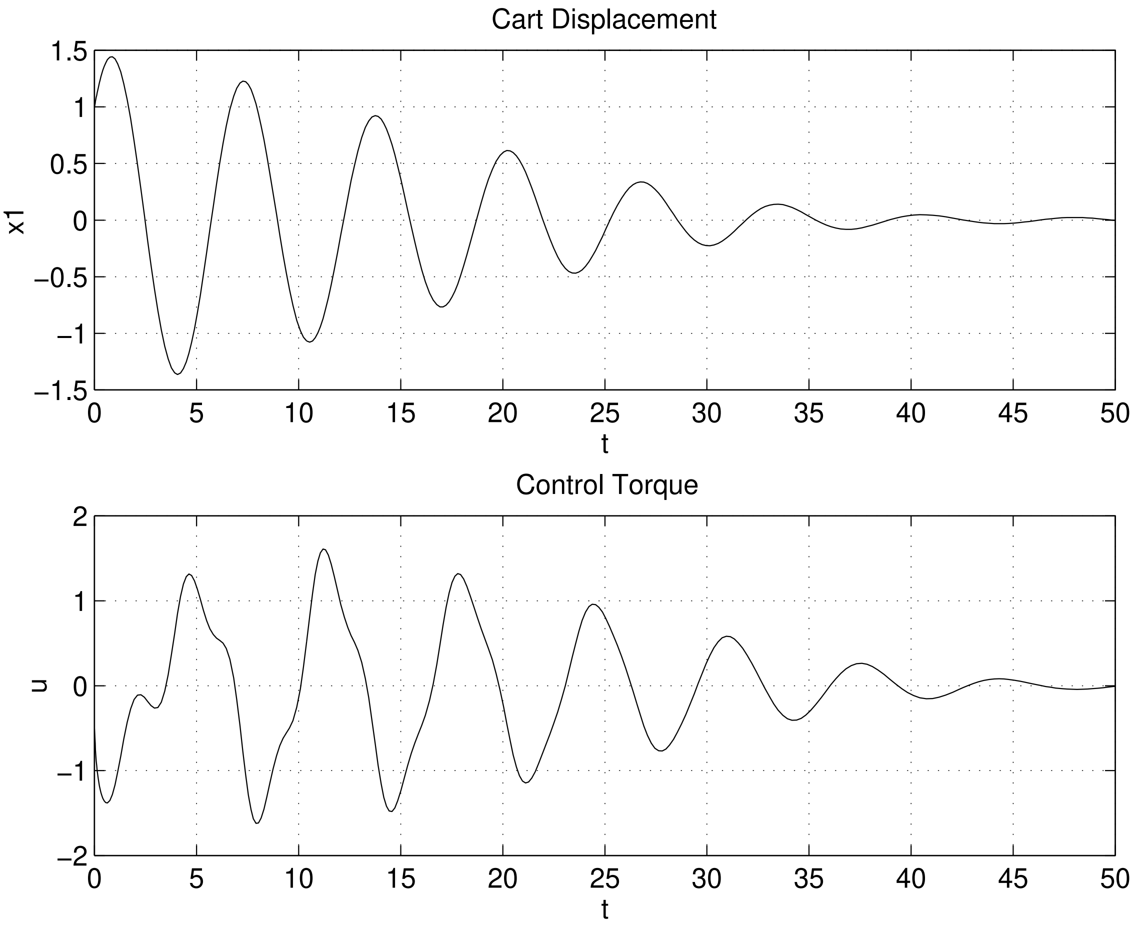

Figure 3: Stabilization with output feedback controller (30).

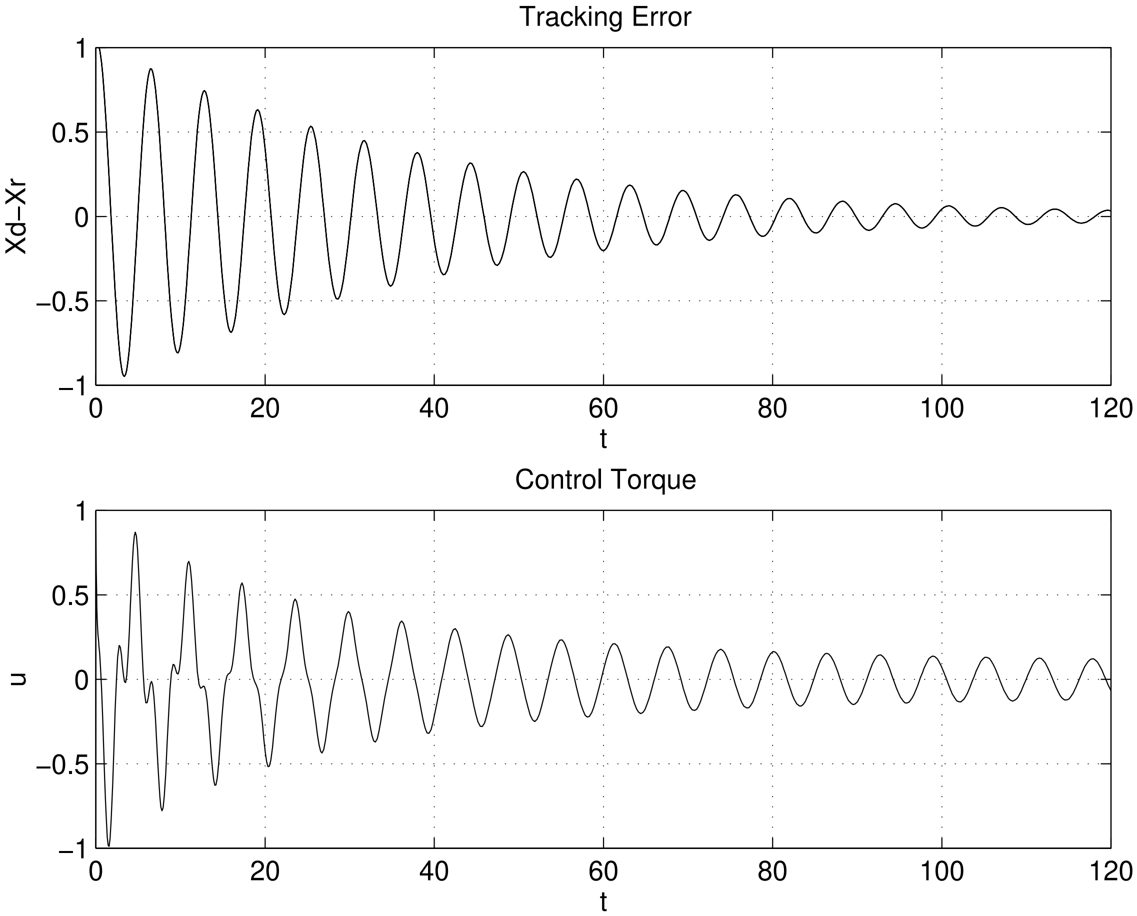

Figure 4: Tracking with output feedback controller (74) and $x_r(t) = \sin t$.

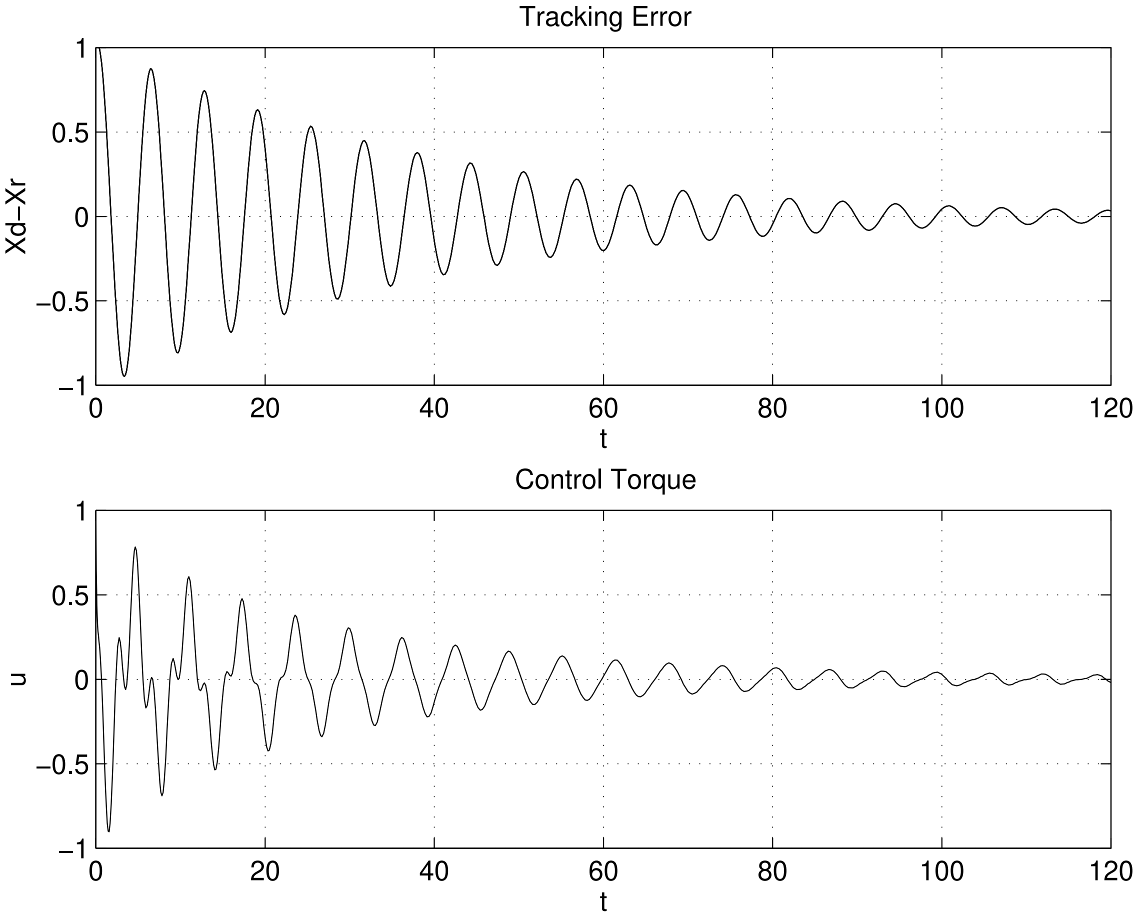

Figure 5: Tracking with output feedback controller (74) and $x_r(t) = 0.1 \sin 2t$.