Global Nonlinear Control of the Ball and Beam System

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Abstract

In this paper, we study the regulation problem for the ball and beam system. Since the system does not have well defined relative degree, exact input-output linearization method may fail. The approximate linearization approach can only be used in a limited region. By representing the system as a fuzzy dynamic model, we give a global control law which is the weighted average of two local stabilization control laws defined over different subspaces. The simulations show that this method gives a much larger stability region than the approximate input-output linearization method.

1 Introduction

The ball and beam system, which can be found in many undergraduate control laboratories, has evoked some interest as an example in nonlinear control recently. Different methods were proposed to control its zero input response and trajectory tracking [1-3]. Because the system has no well defined relative degree, the exact input-output linearization approach may fail. Based on the concept of robust relative degree, Hauser [1] provided the approximate input-output linearization method to solve the problem. Although the controller has good behaviour in a small region, it fails in a larger region. This can be seen easily by observing that the approximate model just omits the high order term (singular point) which will grow in larger regions. So the global control of the ball and beam system is still a challenging problem.

By focusing on the main problem of the ball and beam system, i.e. a singular point exists in a neighbourhood of the origin which results in no well defined relative degree, intuitively we ask why not use the piecewise linearization method to avoid the singular point? In a further study, we can see that if we use the exact input-output linearization in the large region (away from the origin) and use the approximate one in the small region (close to the origin), global control will be available. In this sense, the dynamic fuzzy model-based methods [4-6] are used to represent the system, i.e. membership functions are used to partition the state space and a fuzzy dynamic model to define the different local models in each subspace. Furthermore, after the partition,the high order term in the approximate model can be ensured to be bounded and we can use robust and H_∞ control theory to design a local nonlinear stabilization control law. The fuzzy dynamic model used in this paper extends the linear model form in [4-5] to the nonlinear model form.

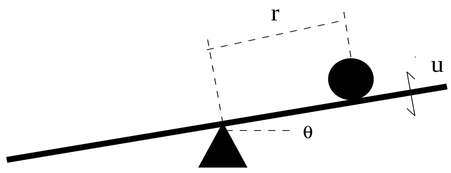

Figure 1: The ball and beam system

This paper is organized as follows. In Section 2, we begin with the nonlinear state space model of the ball and beam system, and its exact and approximate input-output linearization methods are introduced. In Section 3, the fuzzy dynamic model is used to represent the system, and the global control law is given. Then in Section 4, we give a stabilization algorithm and some stability analysis. The simulation results are shown in Section 5. Finally, the paper concludes with some brief remarks in Section 6.

2 Exact and Approximate Input-Output Linearization Method

The familiar ball and beam experiment is depicted in Fig. 1. The beam is made to rotate in a vertical plane by applying a torque at the center of rotation and the ball is free to roll along the beam. We require that the ball remains in contact with the beam and that the rolling occurs without slipping. This imposes a constraint on the rotational acceleration of the beam.

Let $x = (r, \dot{r}, \theta, \dot{\theta})$ be the state of the system and $y = r$ be the output of the system. Then from [1], the system can be represented by the state-space model

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ B(x_1 x_4^2 - G\sin x_3) \\ x_4 \\ 0 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{g(x)} u
$$

$$
y = x_1 \tag{1}
$$

Here the control u is the angular acceleration and the parameters B and G are defined in [1]. The purpose of the regulation problem is to determine $u = u(x)$ such that the closed-loop system output y will converge to zero from given initial conditions.

As is well known, the exact input-output linearization method uses coordinate transformation to derive a feedback control law to make the closed-loop system a linear and controllable one in a neighbourhood of the equilibrium point of interest. It is necessary for the system to have a well defined relative degree at such a point (see [10]). Now let us examine the ball and beam system. Following the usual procedure, we differentiate the output until the input appears

$$
\begin{aligned}
y &= x_1, \\
\dot{y} &= x_2, \\
\ddot{y} &= Bx_1x_4^2 - BG\sin x_3, \\
y^{(3)} &= \underbrace{Bx_2x_4^2 - BGx_4\cos x_3}_{b_1(x)} + \underbrace{2Bx_1x_4}_{a_1(x)} u
\end{aligned}
$$

Because $a_1(x)$ is zero at $x = 0$ but is not identically zero in the neighbourhood of $x = 0$, it is said to have no well defined relative degree at the origin. But if the region of interest is away from any point such that $x_1x_4 = 0$, i.e. $a_1(x) \neq 0$, we could use a control law of the form $u = (-b_1(x) + v_1)/a_1(x)$ to yield a linear input-output system. Here v_1 can be chosen as to make the error system into an exponentially stable linear system. The system after coordinate transformation becomes

$$
\begin{aligned}
(\zeta_1 &= x_1) \\
\dot{\zeta}_1 &= \underbrace{x_2}_{\zeta_2} \\
\dot{\zeta}_2 &= \underbrace{Bx_1x_4^2 - BG\sin x_3}_{\zeta_3} \\
\dot{\zeta}_3 &= \underbrace{Bx_2x_4^2 - BGx_4\cos x_3}_{b_1(x)} + \underbrace{2Bx_1x_4}_{a_1(x)} u \\
&=: v_1(x,u) \\
\dot{\zeta}_4 &= w(x,u)
\end{aligned} \tag{2}
$$

where ζ_4 is the internal dynamics of the system. Here

$$
u = \frac{-b_1(x) + v_1}{a_1(x)} \tag{3}
$$

and

$$
v_1 = \alpha_{12}\zeta_3 + \alpha_{11}\zeta_2 + \alpha_{10}\zeta_1. \tag{4}
$$

Suppose we take $\zeta_4 = x_3$ to complete the transformation. The Jacobian matrix of the transformation is

$$
\frac{\partial \zeta}{\partial x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ Bx_4^2 & 0 & -BG\cos x_3 & 2Bx_1x_4 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

which is nonsingular in the region away from $x_1x_4 = 0$. Then the internal dynamics of (2) is described by

$$
\dot{\zeta}_4 = \varphi(x)(\zeta_3 + BG\sin\zeta_4) \tag{5}
$$

where $\varphi(x) = 1/(x_1x_4)$ is a bounded continuous function with constant sign inside the region S_1 defined below. Because ζ_3 is a continuous function which is forced to go to zero exponentially by v_1 of (4), it follows that ζ_4 is bounded and converges to some equilibrium $\zeta_4 = x_3 = k\pi (k = 0, \pm 1, \pm 2, ...)$. As a result, system (2) does not exhibit internal instability in the region away from $x_1x_4 = 0$.

Due to the lack of well defined relative degree at the origin, Hauser [1] defined the robust relative degree in which the high order $O(x,u)^2$ is neglected from consideration. And then he gave the approximate input-output linearization model as (Approximation 1 in [1])

$$
\begin{aligned}
(\zeta_1 &= x_1) \\
\dot{\zeta}_1 &= \underbrace{x_2}_{\zeta_2} \\
\dot{\zeta}_2 &= \underbrace{-BG\sin x_3}_{\zeta_3} + \underbrace{Bx_1x_4^2}_{\psi(x)} \\
\dot{\zeta}_3 &= \underbrace{-BGx_4\cos x_3}_{\zeta_4} \\
\dot{\zeta}_4 &= \underbrace{-BGx_4^2\sin x_3}_{b_2(x)} + \underbrace{(-BG\cos x_3)}_{a_2(x)} u \\
&=: v_2(x,u)
\end{aligned} \tag{6}
$$

Because the function $\psi(x)$ which is discarded in the approximation is of $O(x)^2$ in a neighborhood of the origin, the chosen

$$
u = \frac{-b_2(x) + v_2}{a_2(x)} \tag{7}
$$

where

$$
v_2 = \alpha_{23}\zeta_4 + \alpha_{22}\zeta_3 + \alpha_{21}\zeta_2 + \alpha_{20}\zeta_1 \tag{8}
$$

can make the approximate system a fully linearization one. Obviously this region cannot be large to ensure $\psi(x) = Bx_1x_4^2$ a uniformly higher order (defined in [1]). Note the equilibrium for x_3 is $k\pi (k = 0, \pm 1, \pm 2, ...)$ and $x_3 = (2k+1)\pi/2$ are singular points which should be avoided.

We can conclude from above that the exact input-output linearization method can only be used in a region away from points such that $x_1x_4 = 0$ and the approximate one is valid only in a small neighbourhood of the origin. In the next section, we consider how to use a fuzzy dynamic model to combine them together to achieve stabilization in a large region.

3 Fuzzy Dynamic Model

Based on the above discussion, we use the following fuzzy model [4-6] to represent the system which includes both approximate inference rules and local analytic models:

$$
\begin{array}{lll}
R_1: & \text{IF} & |\phi(x)| \geq \eta \\
 & \text{THEN} & \dot{x} = f_1(x) + g_1(x)u
\end{array} \tag{9}
$$

$$
\begin{array}{lll}
R_2: & \text{IF} & |\phi(x)| < \eta \\
 & \text{THEN} & \dot{x} = f_2(x) + g_2(x)u + d_2(x)
\end{array} \tag{10}
$$

where η is a positive constant and

$$\phi(x) = x_1x_4$$

$$f_1(x) = f(x) = \begin{bmatrix} x_2 \\ B(x_1x_4^2 - G\sin x_3) \\ x_4 \\ 0 \end{bmatrix},$$

$$g_1(x) = g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$f_2(x) = f(x) + \begin{bmatrix} 0 \\ -Bx_1x_4^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ -BG\sin x_3 \\ x_4 \\ 0 \end{bmatrix},$$

$$g_2(x) = g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, d_2(x) = \begin{bmatrix} 0 \\ -Bx_1x_4^2 \\ 0 \\ 0 \end{bmatrix}.$$

We define the fuzzy sets as

$$
\begin{aligned}
A_1 &= \{x| \quad |\phi(x)| \geq \eta\} \\
A_2 &= \{x| \quad |\phi(x)| < \eta\}
\end{aligned} \tag{11}
$$

And the membership functions
$$\mu_1(x) = \left(1 - \frac{1}{1+\exp(-20(x_1x_4-\eta))}\right)\left(\frac{1}{1+\exp(-20(x_1x_4+\eta))}\right)$$

$$\mu_2(x) = 1 - \mu_1 \tag{12}$$

Then two subspaces are defined in the state space as

$$
\begin{aligned}
S_1 &= \{x| \quad \mu_1 \leq \mu_2\} \\
S_2 &= \{x| \quad \mu_1 > \mu_2\}
\end{aligned} \tag{13}
$$

The characteristic function of each R_l is defined by

$$\lambda_l = \begin{cases} 1 & x \in S_l \\ 0 & \text{otherwise} \end{cases}$$

Note
$$\lambda_1 + \lambda_2 = 1.$$

By using the coordinate transformations (2) and (6) on the systems (9) and (10) respectively, the original system can be represented in the following form,

$$
\begin{array}{lll}
R_1: & \text{IF} & x \in S_1 \\
 & \text{THEN} & \dot{\bar{\zeta_1}} = A_1\bar{\zeta_1} + B_1v_1 \\
 & & \dot{\Psi} = w(\bar{\zeta_1}, \Psi) \\
R_2: & \text{IF} & x \in S_2 \\
 & \text{THEN} & \dot{\bar{\zeta_2}} = A_2\bar{\zeta_2} + B_2v_2 + D\Delta(\bar{\zeta_2})\bar{\zeta_2}
\end{array} \tag{14}
$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \Delta(\bar{\zeta_2}) = \begin{bmatrix} \frac{\zeta_{24}^2}{BG^2\cos^2(\arcsin(\zeta_{23}/(-BG)))} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T.$$

For stabilization of the system (9) and (10),we use the control law

$$u(x) = \begin{cases} u_1(x) & x \in S_1 \\ u_2(x) & x \in S_2 \end{cases} \tag{15}$$

where u_1 and u_2 have the forms of (3) and (7). That is, we use the switching control law to stabilize the ball and beam system; if the states are in the region S_1 (away from the singular point), we use the control law derived from the exact input-output linearization method; whilst in region S_2 (close to the singular point), we use the control law from the approximate linearization method instead. Furthermore, for a smooth combination, we can use the following fuzzy control law which is the average of the individual control laws and weighted by their membership functions

$$u(x) = \mu_1u_2(x) + \mu_2u_1(x). \tag{16}$$

4 Stabilization Algorithm

Due to the fact that coordinate transformations do not change the stability of the original system, we study the stability of system (14) which has a partial linear form instead of (9) and (10). The transformed control law is chosen to have the following form,

$$v(\zeta) = \begin{cases} v_1(\bar{\zeta_1}) & x \in S_1 \\ v_2(\bar{\zeta_2}) & x \in S_2 \end{cases} = \begin{cases} K_1\bar{\zeta_1} & x \in S_1 \\ K_2\bar{\zeta_2} & x \in S_2 \end{cases} \quad (17)$$

The closed-loop system is obtained from the feedback interconnection of fuzzy subsystem (14) and the controller (17), and can be described by,

$$\begin{aligned} R_1: \quad & \text{IF} \quad x \in S_1 \\ & \text{THEN} \quad \dot{\bar{\zeta_1}} = A_{c1}\bar{\zeta_1} && (18) \\ & \qquad\quad \dot{\Psi} = w(\bar{\zeta_1}, \Psi) && (19) \\ R_2: \quad & \text{IF} \quad x \in S_2 \\ & \text{THEN} \quad \dot{\bar{\zeta_2}} = A_{c2}\bar{\zeta_2} + D\Delta(\bar{\zeta_2})\bar{\zeta_2} && (20) \end{aligned}$$

where $A_{c1} = A_1 + B_1K_1$; $\quad A_{c2} = A_2 + B_2K_2$.
Because in the region S_2 we have $|x_1x_4| < \eta$, and x_4 is constrained by the physical fact that rotational acceleration cannot be large to prevent the ball slipping. So $|d_2(x)|$ in (10) is bounded. Therefore $|D\Delta(\bar{\zeta_2})\bar{\zeta_2}|$, which is the form after coordinate transformation, is also bounded, that is,

$$|D\Delta(\bar{\zeta_2})\bar{\zeta_2}| < \bar{\eta} \quad (21)$$

where $\bar{\eta}$ is a positive constant.
Before discussing the stability of the systems (18)-(20), we need the following two theorems.

Theorem 1 *Given the linear system*

$$\dot{\zeta} = A\zeta + Bu + D\Delta(\zeta)\zeta \quad (22)$$

consider the following feedback control law

$$u = K\zeta \quad (23)$$

If the following algebraic Riccati inequality has a positive-definite symmetric solution P for a given $\varepsilon > 0$,

$$A_c^T P + PA_c + \varepsilon PP + \frac{1}{\varepsilon}(D\Delta)^T(D\Delta) < 0 \quad (24)$$

where $A_c = A + BK$ then the system is asymptotically stable.

The proof of Theorem 1 is given in the Appendix.

Remark 1 *Theorem 1 gives a condition to check stability of system (22). For our second fuzzy subsystem (20), we use a controller design method such as pole-placement, LQ design or H_∞ design to obtain the feedback gain K_2, and then if condition (24) holds, we can conclude $K_2\bar{\zeta_2}$ is a stabilizing control law. Otherwise, we should choose another K_2.*

Theorem 2 gives a known theorem for analysing the stability of the dynamic fuzzy systems.

Theorem 2 *(See [6] Theorem 4.2) If there exists a set of continuously differentiable functions $V_l(x,t), (l = 1,2,...,m)$, a set of class $\mathcal{K}$ functions $a_l, b_l, c_l, (l = 1,2,...,m)$ such that*

1. *$\forall t > t_0, \forall x \in S_l$,*
$$a_l(|x|) \le V_l(x,t) \le b_l(|x|) \qquad l = 1,2,...,m \quad (25)$$
2. *$\forall t > t_0, \forall x \in S_l$,*
$$\dot{V}_l(x,t) \le -c_l(|x|) \qquad l = 1,2,...,m \quad (26)$$
3. $$V_l(\tau_{li+1}) \le V_l(\tau_{li}) \qquad l = 1,2,...,m \quad (27)$$
where $\tau_{li}(i = 1,2,...,\overline{T_l}, \tau_{l1} < \tau_{l2} < ... < \tau_{l\overline{T_l}})$ are $\overline{T_l}$ time instants at which the trajectory of the system meets the boundary of S_l.

then the system is asymptotically stable.

Remark 2 *Theorem 2 gives a set of sufficient conditions for asymptotic stability, that is, if there exists a piecewise Lyapunov function such that the subsystems are asymptotically stable in each local region and satisfy the boundary condition (27), then the whole system is asymptotically stable.*

In light of Theorem 2, if the two local control laws in (17) locally stabilize the two subsystems (18)-(19) and (20), and condition (27) is satisfied at the boundary region, then system (18)-(20) is asymptotically stable. We can adjust the linear gain by using the linear pole assignment method to make sure that the Lyapunov functions of the two subsystems decrease fast enough to meet the boundary condition. The simulation results show it is successful.

By now we can conclude our results in the following theorem.

Theorem 3 *For the closed-loop ball and beam system (18)-(20) (equivalently, (14), (9)-(10) and (1)), if the following conditions are satisfied,*

1. *A_{c1} is a stable matrix;*
2. *The algebraic Riccati inequality, $\varepsilon > 0$,*
$$A_{c2}^T P_2 + P_2A_{c2} + \varepsilon P_2P_2 + \frac{1}{\varepsilon}(D\Delta)^T(D\Delta) < 0 \quad (28)$$
has a positive-definite symmetric solution P_2;
3. *The boundary condition holds,*
$$V_l(\tau_{li+1}) \le V_l(\tau_{li}), \qquad l = 1,2 \quad (29)$$

then the system is asymptotically stable. The stabilization control law for (1) is

$$u(x) = \begin{cases} u_1(x) & x \in S_1 \\ u_2(x) & x \in S_2 \end{cases} = \begin{cases} \frac{-b_1(x)+v_1}{a_1(x)} & x \in S_1 \\ \frac{-b_2(x)+v_2}{a_2(x)} & x \in S_2 \end{cases} \quad (30)$$

or

$$u(x) = \mu_1 u_2 + \mu_2 u_1 \tag{31}$$

where $b_1(x)$ and $a_1(x)$ are given in (2), $v_1(x)$ is given in (4), the coefficients $\alpha_{10}, \alpha_{11}, \alpha_{12}$ chosen such that the polynomial $s^3 + \alpha_{12}s^2 + \alpha_{11}s + \alpha_{10}$ is a Hurwitz polynomial; $b_2(x)$ and $a_2(x)$ are given in (6), $v_2(x)$ is given in (8), $K_2 = [\alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}]$, and K_2 chosen such that the algebraic Riccati inequality (28) has a positive-definite symmetric solution P_2.

Proof (Outline) : For the first subsystem (18)-(19), by the Assumption 1, (18) is asymptotically stable and there exists a Lyapunov function $V_1(\bar{\zeta_1})$ satisfying items 1 and 2 of Theorem 2 with $l = 1$. It should be noted that, by contradiction, the internal dynamics (19) (i.e. (5)) tend to the equilibrium $x_3 = k\pi(k = 0, \pm 1, \pm 2, ...)$. So the internal stability of the first subsystem is guaranteed.

For the second subsystem (20), by the Assumption (28) it is asymptotically stable with the quadratic Lyapunov function $V_2 = \bar{\zeta_2}^T P_2 \bar{\zeta_2}$ by adaptation of Theorem 1. Obviously items 1 and 2 of Theorem 2 with $l = 2$ are satisfied.

Because the boundary condition (29) is assumed to be satisfied, in light of Theorem 2, the whole system is asymptotically stable.

Due to the fact that diffeomorphic coordinate transformations do not change the stability of the original system, the asymptotical stability of system (18)-(20) is equivalent to that of the original system (1). The control law after the converse transformation has the form of (30) or (31) which is a weighted average of (30) for smooth connection. □

Remark 3 *Since all points of $x_3 = (2k+1)\pi/2$ located in S_2 are singular points, there exists the possibility that x_3, the uncontrollable state component in S_1, does not reach nearby its equilibrium $x_3 = k\pi$ when x enters the region S_2. This fact prevents us from claiming global asymptotic stability. However, the stability domain in Theorem 3 should be much larger than in [1].*

5 Simulation results

In our simulation,let $B = 0.7143$,$G = 9.81$ as defined in [1]. We use membership functions as (12) with $\eta = 1$ (shown in Figure 2). We try two sets of initial conditions and compare the regulation results by our method ('Method 2' as mentioned in the following) with the approximate input-output linearization method by Hauser's in [1] ('Method 1' as mentioned in the following).

For the first set of initial conditions $x(0) = [0.6, -0.1, 0.6, 0.1]$, $[-0.6, 0.1, -0.6, -0.1]$, $[2.4, -0.1, 0.6, 0.1]$, $[-2.4, 0.1, -0.6, -0.1]$, Figure 3 and Figure 4 demonstrate the responses. We can see that they are both stable by Method 1 and 2, but Method 2 has a better response.

For the second set of four initial conditions $x(0) = [9, -0.5, 0.345, 0.5]$, $[-9, 0.5, -0.345, -0.5]$, $[18, -0.5, 0.345, 0.5]$, $[-18, 0.5, -0.345, -0.5]$, Method 1 cannot stabilize the system by the first one $x(0) = [9, -0.5, 0.345, 0.5]$ as shown in Figure 5, and cannot stabilize the other three either. Whilst Method 2 can stabilize the system by the four initial conditions as shown in Figure 6.

6 Conclusions

In this paper, by analyzing the limitation of exact input-output linearization and the approximate linearization method for the ball and beam system, we represent a new approach to globally stabilize its regulation problem. A fuzzy dynamic model is used to represent the system, and membership functions are to partition the state space. Then we use piecewise Lyapunov function to study the stability problem. The final control law is a nonlinear control law which is composed of two local stabilization control laws derived from their linearization models defined over each subspace. The simulation results show that the stability region by our method is much larger than that of the approximate input-output linearization method given by Hauser in [1].

Appendix

In the proof of Theorem 1,the following preliminary results will be used.

Lemma 1 *(See [9] for proof)For any constants $\varepsilon > 0$ and any matrix X and Y with appropriate dimensions,we have*

$$X^T Y + Y^T X \leq \varepsilon X^T X + \frac{1}{\varepsilon} Y^T Y \tag{32}$$

Proof of Theorem 1: Choose Lyapunov function,

$$V = \zeta^T P \zeta$$

Its derivative is given by

$$\begin{aligned} \frac{dV}{dt} &= \dot{\zeta}^T P \zeta + \zeta^T P \dot{\zeta} \\ &= (A_c\zeta + D\Delta\zeta)^T P\zeta + \zeta^T P(A_c\zeta + D\Delta\zeta) \\ &= \zeta^T(A_c^T P + PA_c)\zeta + \zeta^T[P(D\Delta) + (D\Delta)^T P]\zeta \end{aligned} \tag{33}$$

In light of Lemma 1,the second term in (37) becomes

$$P(D\Delta)+(D\Delta)^T P \le \varepsilon P^T P + \frac{1}{\varepsilon}(D\Delta)^T(D\Delta) \quad (34)$$

Substituting (38) into (37),we have

$$\frac{dV}{dt} \le \zeta^T[A_c^T P + PA_c + \varepsilon P^T P + \frac{1}{\varepsilon}(D\Delta)^T(D\Delta)]\zeta \quad (35)$$

If there exists a positive-definite symmetric matrix P satisfying the following algebraic Riccati inequality

$$A_c^T P + PA_c + \varepsilon PP + \frac{1}{\varepsilon}(D\Delta)^T(D\Delta) < 0 \quad (36)$$

then

$$\frac{dV}{dt} < 0$$

By a standard Lyapunov stability theorem [11], the system is asymptotically stable.□

References

[1] J. Hauser, S. Sastry and P. Kokotovic, "Nonlinear control via approximate input-output linearization:the ball and beam example", *IEEE Trans. Automat. Contr.*, vol.37, pp. 392-398, 1992

[2] L. X. Wang and J. M. Mendel, "Fuzzy basic functions, universal approximation, and orthogonal least-squares learning", *IEEE Tran. on neural Networks*, vol.3, pp. 807-814, 1992

[3] P. Dower, "Nonlinear state feedback H_∞ control of the ball and beam system", Honours Thesis, Dept. of Electrical and Computer Engineering, Univ. of Newcastle, Australia, 1993

[4] S. G. Cao and N. W. Rees, "Identification of dynamic fuzzy models", *Fuzzy sets and systems 74*, pp. 307-320, 1995

[5] S. G. Cao, N. W. Rees and G. Feng, "Quadratic stability analysis and design of continuous-time fuzzy control systems", *Int. J. Systems Science*, vol.27, pp. 193-203, 1996

[6] S. G. Cao, N . W. Rees and G. Feng, "Lyapunov-like stability theorems for continuous-time fuzzy control systems", to appear in *Int. J. Control*, 1996

[7] A. J. van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control", *IEEE Trans. Automat. Contr.*, vol.37, pp. 770-784, 1992

[8] B. Kuipers and K. J. Astrom, "The composition and validation of heterogeneous control laws", *Automatic*, vol.30, pp. 233-249, 1994

[9] I. R. Petersen, "A stabilization algorithm for a class of uncertain linear system", *Systems & Control Letters 8*, pp. 351-357

[10] A. Isidori, *Nonlinear Control Systems*, 3rd ed., London : Springer-Verlag, 1995

[11] W. Hahn, *Stability of motion*, Springer-Verlag, Berlin, 1967

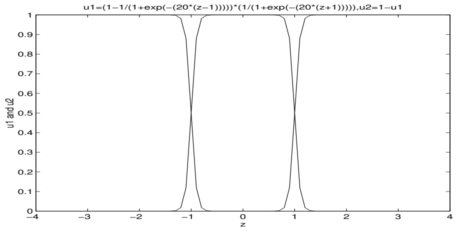

Figure 2: Membership functions μ_1 and μ_2

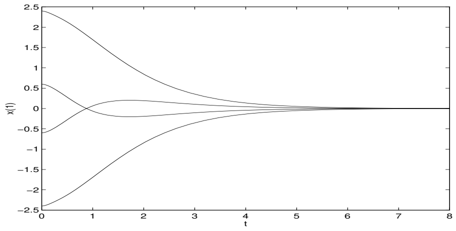

Figure 3: Method 1 and 1st set of $x(0)$

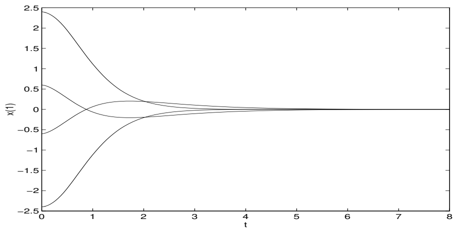

Figure 4: Method 2 and 1st set of $x(0)$

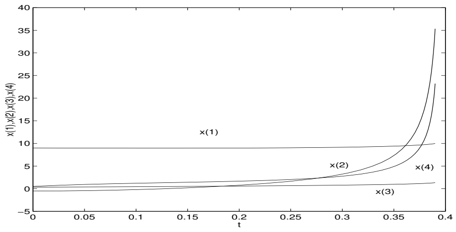

Figure 5: Method 1 and one of 2nd set of $x(0)$

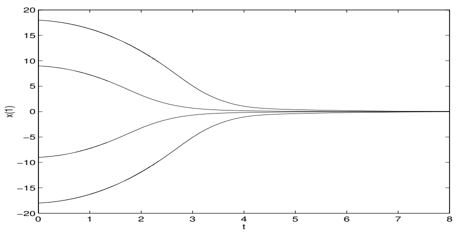

Figure 6: Method 2 and 2nd set of $x(0)$