Fuzzy Dissipativity Design for Global Disturbance Rejection for the Nonlinear Benchmark Problem

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Abstract— **In this paper, we study the fuzzy dissipativity design method and apply it to the nonlinear benchmark problem. By representing the nonlinear system with a dynamical fuzzy model and then rendering each local subsystem dissipative, a global control law can be obtained by smooth aggregation of the local controllers. This method avoids solving the global nonlinear Hamilton-Jacobi inequality but achieves global disturbance rejection for the closed loop system.**

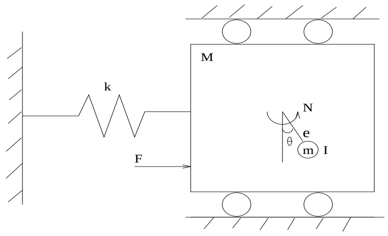

Fig. 1. Rotational/translational proof mass actuator (RTAC)

I. Introduction

In recent years, a promising direction in control is the extension of the linear H_∞ control theory to the nonlinear case. By solving the Hamilton-Jacobi inequality (HJI), which is a nonlinear partial differential inequality, the nonlinear H_∞ control problem can be solved by constructing a feedback controller rendering the closed-loop system dissipative from the exogenous input (disturbances and references) to the output variables [3]. However until now, few practical solution methods are known for solving the HJI. One particular approach mostly used is the Lukes' method [6] which is based on a series expansion of the problem and leads to a locally approximate solution. A global analytical solution is not feasible except for simple problems. Therefore, from a practical engineering viewpoint, the global nonlinear H_∞ control problem is still a challenging one to study [1].

In light of the success of gain scheduling and fuzzy control in many engineering applications, we consider a fuzzy dissipativity design method for the global nonlinear H_∞ control problem. The main idea is to partition the state space into a set of local regions and represent the nonlinear system by a dynamical fuzzy model [4] which includes a set of local simple subsystems defined on the local regions. Then in each local region the storage function rendering the closed loop subsystem dissipative is much easier to find corresponding to the simple subsystem structure (and can even be quadratic ones corresponding to local linear subsystems [5]). Thus, a piecewise differentiable storage function is available and a set of local control actions are obtained. Membership functions are used to smoothly aggregate the local control actions into a global control one.

The nonlinear benchmark problem which was proposed for examining the different nonlinear design techniques has been studied by several authors and with different techniques [7]-[9]. Stabilising control laws have been obtained by backstepping design [7] and passivity design [8]. But the disturbance rejection problem of this system is a more challenging one. In [9], by applying the series expansion method for solving the nonlinear HJI, a higher order term controller is given rendering the system dissipative from the disturbance to the performance output in the L_2-gain sense. However, a simulation comparison with the linear controller (which is the linear H_∞ controller obtained from the linearized problem) shows that the performance improvement is not significant and the extension of the region of attraction is limited. The contribution of the present paper is to use the fuzzy dissipativity method to design a disturbance rejection controller which is effective globally for the nonlinear benchmark problem.

The notation used in this paper is fairly standard. $|x|$ is the Euclidean norm of the vector x. $\|d\| := \text{ess.sup}.\{|d(t)|, t \geq 0\}$.

II. Problem Statement

The basic system which is shown in Figure 1 was stated in [7]. The state space model is represented as

$$\dot{x} = f(x) + g(x)u + k(x)d \tag{1}$$

where u is the input and d is the disturbance of the system. The vectors $f(x), g(x), k(x)$ are given by

$$f(x) = \begin{bmatrix} x_2 \\ \frac{-x_1+\varepsilon x_4^2 \sin\ x_3}{1-\varepsilon^2 \cos^2\ x_3} \\ x_4 \\ \frac{\varepsilon \cos\ x_3(x_1-\varepsilon x_4^2 \sin\ x_3)}{1-\varepsilon^2 \cos^2\ x_3} \end{bmatrix},$$

$$g(x)=\begin{bmatrix} 0 \\ \frac{-\varepsilon\cos\ x_3}{1-\varepsilon^2\cos^2\ x_3} \\ 0 \\ \frac{1}{1-\varepsilon^2\cos^2\ x_3}\end{bmatrix},\ k(x)=\begin{bmatrix} 0 \\ \frac{1}{1-\varepsilon^2\cos^2\ x_3} \\ 0 \\ \frac{-\varepsilon\cos\ x_3}{1-\varepsilon^2\cos^2\ x_3}\end{bmatrix}$$

with ε being a constant parameter in $(0,1)$. A typical value $\varepsilon=0.1$ has been used in simulations.

By partial feedback linearization, a simple system form was given as ([7])

$$\begin{aligned} \dot{z_1} &= z_2 \\ \dot{z_2} &= -z_1+\varepsilon\sin\ z_3+d \\ \dot{z_3} &= z_4 \\ \dot{z_4} &= v+\frac{-\varepsilon\cos\ z_3}{1-\varepsilon^2\cos^2\ z_3}d \end{aligned} \tag{2}$$

The corresponding feedback transformation is

$$\begin{aligned} v &= \frac{1}{1-\varepsilon^2\cos^2\ x_3}(\varepsilon x_1\cos\ x_3-\varepsilon^2 x_4^2\cos\ x_3\sin\ x_3) \\ &\quad +\frac{1}{1-\varepsilon^2\cos^2\ x_3}u \\ &= \alpha(x)+\beta(x)u \end{aligned} \tag{3}$$

which is nonsingular.

The global disturbance rejection problem studied in this paper is defined as follows.

Global Disturbance Rejection Problem: Assuming that all the states in (2) are available, find a memoryless feedback control law $v=k(z)$ such that for each initial condition $z_0\in\Re^4$ and a class of disturbances d satisfying $||d||\le\sigma(z_0)$ ($\sigma(\cdot)$ is a continuous function), the closed loop system meets the following dissipation inequality

$$\int_0^T\{|y(t)|^2-\gamma^2|d(t)|^2\}dt\le\phi(z_0)\quad\forall T\ge 0 \tag{4}$$

where $\phi(z_0)$ is a continuous function and y denotes the performance output

$$y=\begin{bmatrix} Hz \\ u\end{bmatrix}$$

and γ is a design parameter. Further, there exists a time t_0 such that

$$\int_{t_0}^T\{|y(t)|^2-\gamma^2|d(t)|^2\}dt\le 0\quad\forall T\ge t_0 \tag{5}$$

Remark 1: The global disturbance rejection problem is interpreted here as a problem for arbitrary initial conditions but for a particular class of L_2-disturbances. Property (4) requires the closed loop system to have L_2 gain $\le\gamma$ from d to y.

In the next section, we discuss the fuzzy dissipativity design for the general nonlinear system. And then in Section 4, we show how the fuzzy dissipativity design method can be used in the global disturbance rejection problem.

III. Fuzzy Dissipativity Design

Consider the general nonlinear system described by a state-space model of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t),u(t),d(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{6}$$

where $x(t)\in\Re^n$ is the state vector, $u(t)\in\Re^m$ are the control inputs, $d(t)\in\Re^p$ are the exogenous inputs (disturbance to be rejected or reference to be tracked) belonging to $L_2(0,\infty)$, $y\in\Re^q$ are the to-be-controlled outputs. Without loss of generality, we assume the existence of a fixed equilibrium $(x_0,u_0,d_0)=(0,0,0)$ with $f(x_0,u_0,d_0)=0, h(x_0)=0$.

Because the above model contains global nonlinearities, it is very difficult to design a global control law based on it. It is generally understood that it is usually possible to express such a system as a number of simple systems (particularly linear systems) in a number of local regions.

We represent the system as a dynamical fuzzy model ([4]-[5]) which includes a set of local subsystems together with the fuzzy set representations

$$\begin{aligned} R_l:\quad &\text{IF}\quad x_1\ \text{is}\ F_1^l\quad\text{AND}\quad x_2\ \text{is}\ F_2^l \\ &\qquad\text{AND}\quad ...\quad\text{AND}\quad x_n\ \text{is}\ F_n^l \\ &\text{THEN}\quad \dot{x}=f_l(x,u,d) \\ &\qquad\quad y=h_l(x) \end{aligned} \tag{7}$$

where R_l denotes the l-th approximation inference rule. Also

$$F^l=\prod_{i=1}^n F_i^l,\quad l=1,2,...,\bar{m}$$

denotes the fuzzy sets in state space, (f_l,h_l) represent the l-th local model of the nonlinear system (6), and $\bar{m}$ is the number of approximation inference rules.

By applying the normalized membership function of the inferred fuzzy set F^l as $\mu_l(x(t))$ satisfying

$$\sum_{l=1}^{\bar{m}}\mu_l=1, \tag{8}$$

the fuzzy subspaces can be defined as follows,

$$S_l=\{x|\mu_l(x)>\mu_i(x),\ \ i=1,2,...,\bar{m},\ \ i\ne l\} \tag{9}$$

$$l=1,2,...,\bar{m}$$

The boundary of S_l is denoted by

$$\partial S_l=\{x|\mu_l(x)=\mu_i(x),\quad i=1,2,...,\bar{m},\ \ i\ne l\} \tag{10}$$

The characteristic function of S_l is

$$\lambda_l=\begin{cases} 1 & x\in S_l \\ 0 & x\notin S_l\end{cases},\qquad\sum_{l=1}^{\bar{m}}\lambda_l=1. \tag{11}$$

So the dynamical fuzzy model of system (6) becomes

$$\begin{aligned} R_l: \quad & \text{IF} \quad x \in S_l \\ & \text{THEN} \quad \dot{x} = f_l(x,u,d) \\ & \qquad\quad\; y = h_l(x) \end{aligned} \tag{12}$$

Since system (6) is decomposed into several subsystems in (12) on several different local regions, we will use the following piecewise differentiable storage function for stability analysis,

$$V = \sum_{l=1}^{\bar{m}} \lambda_l V_l \tag{13}$$

where λ_l satisfies (11). That means,

$$V = V_l, \quad \text{when} \quad x \in S_l. \tag{14}$$

The notations used in this paper are similar to that in [5] and restated as follows. Let $\tau_i, i = 1, 2, ..., T^*$ be T^* time instants at which the states of the subsystems in (12) meets the boundary of one of $S_l, l = 1, 2, ..., \bar{m}$. Thus, all discontinuous points of $V(x(t))$ will happen at the time instants τ_i. For example, suppose that $x(\tau_i^-) \in S_l$, then we may have

$$V(x(\tau_i^-)) = V_l(x(\tau_i)), \;\; V(x(\tau_i^+)) = V_j(x(\tau_i)), \;\; j = l+1. \tag{15}$$

where $V(x(\tau_i^-))$ denotes the left limit of $V(x(\tau_i))$ and $V(x(\tau_i^+))$ denotes the right limit of it. Thus, it is clear that the state x and the output y of the system jump at discrete instants of the time τ_i and be left continuous but right discontinuous with respect to S_l at the time instance $t = \tau_i$.

The global dissipativity of the dynamical fuzzy system (12) can be guaranteed by the following theorem.

Theorem 1: For the dynamical fuzzy system (12), if there exists a piecewise differentiable storage function (13) such that in each local subspace the local dissipativity inequality is satisfied, i.e.

$$V_l(x(t_1)) - V_l(x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), y(t))dt \tag{16}$$

where $s(u(t), y(t))$ is a given supply rate, $t_1 > t_0$, and the boundary conditions are met

$$V(x(\tau_i^+)) \leq V(x(\tau_i^-)), \tag{17}$$

then the system is globally dissipative with the supply rate $s(u(t), y(t))$.

The proof of Theorem 1 is given in [11].

IV. The Nonlinear Benchmark Problem

In this section, we apply the above discussed fuzzy dissipativity design methodology to the global disturbance rejection of the nonlinear benchmark problem stated in Section 2.

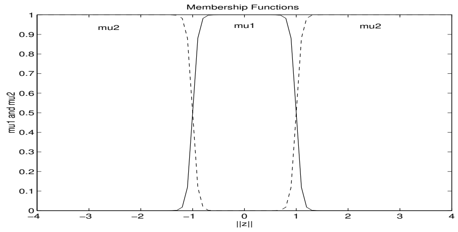

Fig. 2. Membership functions μ_1 and μ_2 ($\eta = 1$)

A. Fuzzy Dynamic Model

We define the fuzzy sets as

$$\begin{aligned} A_1 &= \{z| \quad |z| \geq \eta\} \\ A_2 &= \{z| \quad |z| < \eta\} \end{aligned} \tag{18}$$

where η is a positive constant,

$$|z| = \sqrt{z_1^2 + z_2^2 + z_3^2 + z_4^2}$$

and the membership functions

$$\mu_1(z) = (1 - \frac{1}{1 + exp(-20(|z| - \eta))})(\frac{1}{1 + exp(-20(|z| + \eta))})$$

$$\mu_2(z) = 1 - \mu_1$$

which are shown in Fig. 2.

Then the two subspaces can be defined as

$$\begin{aligned} S_1 &= \{z| \quad \mu_1 \geq \mu_2\} \\ S_2 &= \{z| \quad \mu_1 < \mu_2\} \end{aligned} \tag{19}$$

We represent system (2) by the dynamical fuzzy model

$$\begin{aligned} R_1: \quad & \text{IF} \quad z \in S_1 \\ & \text{THEN} \quad \dot{z} = f_1(z) + g_1 u + k_1 d \\ & \qquad\quad\; y = h_1(z) \end{aligned} \tag{20a}$$

$$\begin{aligned} R_2: \quad & \text{IF} \quad z \in S_2 \\ & \text{THEN} \quad \dot{z} = f_2(z) + g_2 u + k_2 d \\ & \qquad\quad\; y = h_2(z) \end{aligned} \tag{20b}$$

where

$$f_1(z) = f_2(z) = \begin{bmatrix} z_2 \\ -z_1 + \varepsilon \sin\; z_3 \\ z_4 \\ 0 \end{bmatrix},$$

$$g_1(z) = g_2(z) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$k_1(z) = k_2(z) = \begin{bmatrix} 0 & 1 & 0 & \frac{-\varepsilon \cos\; z_3}{1-\varepsilon^2 \cos^2\; z_3} \end{bmatrix}^T,$$

$$h_1(z) = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, h_2(z) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

B. Local Control Law Design

Now we discuss the local controller design.

In S_2, which denotes the global area far away from the equilibrium, we design a global passivity-based control law. Since system (20b) (d=0) has relative degree 1 and is weakly minimum phase, it is feedback equivalent to a passive system ([2]). Choosing the storage function as

$$V_2(z) = \frac{1}{2}k_0(z_1 - \varepsilon \sin\, z_3)^2 + \frac{1}{2}k_0 z_2^2 + \frac{1}{2}k_1 z_3^2 + \frac{1}{2}z_4^2 \quad (21)$$

where k_0, k_1, k_2 are constant parameters, a feedback control law can be constructed as follows rendering the closed loop system strictly passive with the supply rate $s = y^T u$,

$$v_2 = k_0(z_1 - \varepsilon \sin\, z_3)(\varepsilon \cos\, z_3) - k_1 z_3 - k_2 z_4. \quad (22)$$

In the simulation we use the parameters $(k_0, k_1, k_2) = (10, 1, 0.5)$ ([8]).

In S_1, which denotes the local area around the equilibrium, we use the nonlinear H_∞ controller obtained by Lukes' series expansion method ([6],[9]). That is, represent f_1, g_1, k_1, h_1, V_1 in their Taylor series expansion forms,

$$f_1(z) = \sum_{i=1}^{\infty} f_1^{[i]}(z), g_1(z) = \sum_{i=0}^{\infty} g_1^{[i]}(z),$$

$$k_1(z) = \sum_{i=0}^{\infty} k_1^{[i]}(z), h_1(z) = \sum_{i=1}^{\infty} h_1^{[i]}(z),$$

$$V_1(z) = \sum_{i=2}^{\infty} V_1^{[i]}(z).$$

Substitute them into the Hamilton-Jacobi equation,

$$\frac{\partial V_1}{\partial z}(z) f_1(z) + \frac{1}{2}\frac{\partial V_1}{\partial z}(z)\left[\frac{1}{\gamma^2}k_1(z)k_1^T(z) - g(z)g^T(z)\right] \frac{\partial^T V_1}{\partial z}(z) + \frac{1}{2}h_1^T(z)h_1(z) = 0 \quad (23)$$

and equate the kth ($k \geq 2$) order term to zero. By solving the resulting linear system equations for the unknown coefficient vector of $V_1^{[k]}$, the gradient of the expansion for $V_1^{[k]}$ is available. Starting from $k = 2$, the higher order solution can be obtained recursively. The controller is constructed by

$$v_1^{[k]} = -\sum_{i=0}^{k-1} g_1^{[i]T}(z)\frac{\partial^T V_1^{[k-i+1]}}{\partial z}(z) \quad (24)$$

C. Global Control Law

After two local control laws are constructed, the globally defined control law is obtained by

$$v(z) = \begin{cases} v_1(z) & z \in S_1 \\ v_2(z) & z \in S_2 \end{cases} \quad (25)$$

which is a switching control law or

$$v(z) = \mu_1 v_1(z) + \mu_2 v_2(z) \quad (26)$$

which is the smooth connection of the two local control laws.

The control law for the original system model (1) is

$$u = \mu_1 u_1 + \mu_2 u_2 \quad (27)$$

where

$$u_i = \frac{v_i - \alpha(x)}{\beta(x)}, \qquad i = 1, 2.$$

and $\alpha(x), \beta(x)$ are defined in (3).

D. Stability Anaylsis

Proposition 1 is the direct application of Theorem 2 of [10] to the system (2).

Proposition 1: If the system (2) ($d = 0$) is globally asymptotically stable with $v = v_2(z)$, then there exist a function β of class $\mathcal{KL}$, a function γ of class $\mathcal{K}$, and a continuous function $\sigma : \Re_+ \to \Re_+$, $\sigma(s) \neq 0$ if $s \neq 0$, such that, for each initial state z_0 and for each measurable essentially bounded $d(\cdot)$ for which

$$\|d\| \leq \sigma(|z_0|), \quad (28)$$

the solution of (2) exists for each $t \geq 0$ and satisfies

$$|z(t)| \leq \beta(|z_0|, t) + \gamma(\|d\|). \quad (29)$$

Furthermore, there exists a constant time $t_0 > 0$ and a compact set S such that $z(t) \in S$ for all $t \geq t_0$.

Assumption 1: The above compact set S is a subset of W which is the domain of attraction of system (2) with the nonlinear H_∞ controller $v = v_1(z)$. That is, $S \subset W$.

Theorem 2: Suppose Assumption 1 is satisfied, then the global disturbance rejection problem is solvable.

The proof of Theorem 2 is now straight forward. As long as the stabilizing controller v_2 renders the state z into the domain W, the nonlinear H_∞ controller v_1 keeps it inside the domain and tends to a residual set with certain L_2 gain.

V. Simulations

In the simulations, we compare the stability region obtained by our fuzzy dissipativity control law with that by the nonlinear H_∞ control law with the existence of disturbance $d = 0.1 \sin 0.1t$. The closed loop response for two sets of initial conditions are presented here. For the first set of initial conditions $x_0 = [5, 0, 0, 0]$, Fig. 3 shows that the nonlinear H_∞ controller cannot stabilize the system. As a comparation, the response by the fuzzy dissipativity controller is shown in Fig. 4. And in Fig. 5 the closed-loop response with another initial condition $x_0 = [10, 5, -2, 2]$ is demonstrated. The simulations suggest that the fuzzy dissipativity controler acheives a much larger region of attraction.

VI. Conclusions

We have presented a new nonlinear control design method in this paper based on fuzzy and dissipativity-based control. By partitioning the state space and representing the general nonlinear system by a dynamical fuzzy model, the global nonlinear system can be replaced by a group of local subsystems which have simple subsystem structures. Since it is much easier to find local control actions for local simple subsystems, the global control law can be constructed by the smooth aggregation of the local ones. The stability of the closed loop system is studied using a piecewise differentiable storage function and it is shown that if some boundary conditions are satisfied the stability will be guaranteed. Applying the method to the task of global disturbance rejection for the nonlinear benchmark problem, we use the control law which is the weighted average of a global passivity-based controller and a local H_∞ controller. The transient response may not be as good as that by the global H_∞ controller, which cannot be obtained, but the steady states will have the same L_2 norm bound. It is believed that the fuzzy dissipativity method gives a practical way for effectively solving the global nonlinear H_∞ problem. The simulations verify that for the benchmark problem, the fuzzy dissipativity controller achieves a much larger region of attraction than that of the nonlinear H_∞ controller obtained by solving the HJI in an approximate method.

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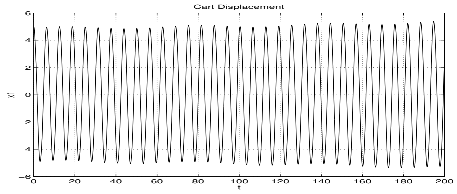

Fig. 3. Nonlinear H∞ controller with $x_0 = [5, 0, 0, 0]$

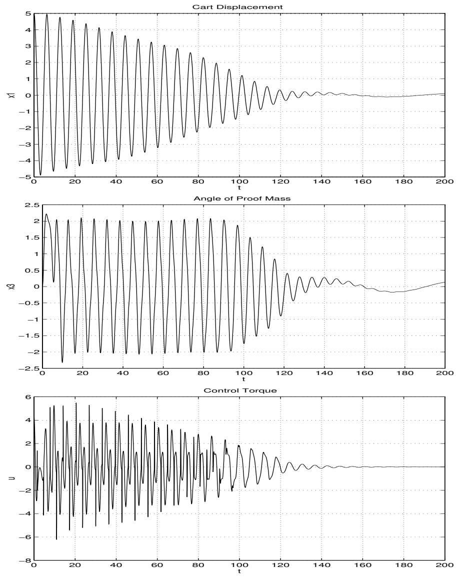

Fig. 4. Fuzzy dissipativity controller with $x_0 = [5, 0, 0, 0]$

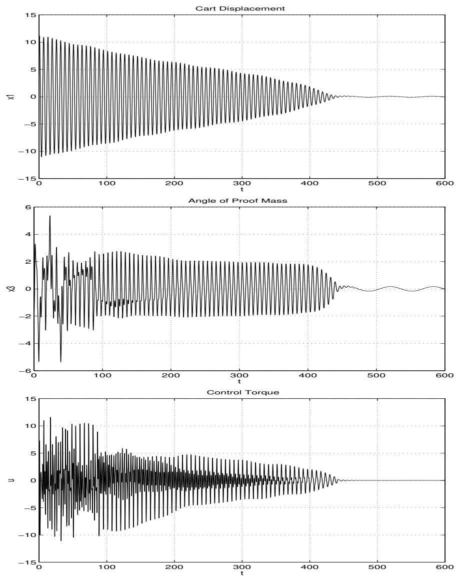

Fig. 5. Fuzzy dissipativity controller with $x_0 = [10, 5, -2, 2]$