Assignment 10

This homework is due Wednesday, December 7th.

There are total 38 points in this assignment. 34 points is considered 100%. If you go over 34 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations (unless stated otherwise in the problem). Bare answers will not earn you much.

This assignment covers sections 7.1, 7.3, 7.4 in O’Neill.

1. (Part of 7.1.1) Let $M$ be a region of $\mathbb{R}^2$ with inner product $\langle v, w \rangle = \frac{v \cdot v}{h^2}$, $h > 0$. Show that:

   (a) [2pt] Angle between vectors $v, w$ in the sense of $\langle \cdot, \cdot \rangle$ coincides with the angle between them in the sense of Euclidean dot product $\cdot$. (Note: hence, the terminology: such geometric surface is called conformal with ruler function $h$.)

   (b) [2pt] The speed of curve $\alpha = (\alpha_1, \alpha_2)$ is $\sqrt{\alpha_1'^2 + \alpha_2'^2}/h(\alpha)$.

   (c) [2pt] $hU_1, hU_2$ is a frame field with dual 1-forms $du/h, dv/h$.

2. (Part of 7.1.2, 7.3.1) The Poincaré half-plane is the upper half plane $v > 0$ in $(u, v)$-plane $\mathbb{R}^2$ with metric $\langle v, w \rangle = \frac{v \cdot w}{v^2}$. If $\alpha$ is a curve $(r \cos t, r \sin t)$, $0 < t < \pi$, with constant $r > 0$,

   (a) [3pt] show the speed of $\alpha$ is $\csc t$. (Note: in particular, does not depend on $r$.)

   (b) [2pt] Deduce that although the Euclidean length of $\alpha$ is $\pi r$, its Poincaré length is infinite.

   (c) [3pt] Show that the connection form of the frame field $E_1 = vU_1, E_2 = vU_2$ is $\omega_{12} = du/v = \theta_1$.

   (d) [3pt] Express velocity and acceleration of the curve $\alpha$ in terms of $E_1, E_2$ ($\alpha$ as defined above).

   (e) [3pt] Express velocity and acceleration of for the Euclidean straight line $\beta(t) = (ct, st)$ in terms of $E_1, E_2$, $(c, s$ are constants such that $c^2 + s^2 = 1, and values of $t$ are such that $st > 0).$ Verify calculations by checking that $\langle \beta', \beta'' \rangle = 2(\beta', \beta'')$. 

3. (7.1.4) (Coordinate definition of a metric.)

   (a) [4pt] If $a, b, c$ are numbers such that $a > 0, c > 0, ac - b^2 > 0$, then the formula

   $$\langle v, w \rangle = av_1w_1 + b(v_1w_2 + v_2w_1) + cw_2w_2$$

   defines an inner product on $\mathbb{R}^2$. (Hint: To prove positive definiteness, consider $(v_1\sqrt{a} \pm v_2\sqrt{c})^2 \geq 0$ and compare it to the required inequality.)
(b) [4pt] Let \( x : D \to M \) be a coordinate patch in an abstract (generally speaking, not geometric) surface \( M \). Given differentiable functions \( E,F,G : D \to \mathbb{R} \) such that

\[
E > 0, \quad G > 0, \quad EG > F^2,
\]

prove that there is a unique metric \( \langle \cdot, \cdot \rangle \) on the image of \( x \) such that

\[
E = \langle x_u, x_u \rangle, \quad F = \langle x_u, x_v \rangle, \quad E = \langle x_v, x_v \rangle.
\]

(4) [3pt] (7.4.2) If \( \gamma_v \) is the unique geodesic in \( M \) with initial velocity \( v \), show that for any number \( c \), \( \gamma_{cv} = \gamma_v(ct) \) for all \( t \).

(5) (7.4.6) In the projective plane \( P(r) \) of radius \( r \) (sphere \( \Sigma(r) \) of radius \( r \) with identified antipodal points), prove:

(a) [3pt] The geodesics are simple (non self-intersecting) closed curves of length \( \pi r \).

(b) [2pt] There is a unique geodesic route through any two distinct point.

(c) [2pt] Two distinct geodesic routes meet in exactly one point.

(Hint: Within “small” patch (here, “small” means that image of patch fits inside a hemisphere), geometries of sphere \( \Sigma(r) \) and \( P(r) \) are the same.)