Finite index subgroups of limit groups

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Subgroups of a Lyndon's free group $F^{\mathbb{Z}[t]}$

Denote $F_0 = F(X)$, a free group on a free generating set X.

We say that for $w \in F_0$, its normal form is a reduced word in generators equal to w: $\pi(w) = \bar{w}$.

Suppose $F_0 < \ldots < F_{n-1}$ and the corresponding normal form π are constructed.

Choose $u \in F_{n-1}$ (not any u is good, but we skip that moment).

Then F_n is generated by F_{n-1} and formal expressions of the form $\{u^\alpha | \alpha \in \mathbb{Z}[t]\}$.

Elements $u^\alpha, \alpha \in \mathbb{Z}[t] - \mathbb{Z}$ are thought to be "infinite powers" of u.

That is, every element of F_n can be viewed as a word of the form

$$w = w_1 u^{\alpha_1} w_2 u^{\alpha_2} \cdots w_m u^{\alpha_m} w_{m+1},$$

where $m \in \mathbb{N}$, $w_i \in F_{k-1}$, and $\alpha_i \in \mathbb{Z}[t] - \mathbb{Z}$.

Note that $\alpha_1, \alpha_2, ..., \alpha_m$ are not defined uniquely.

Example

Let $F_0 = F(x,y), u = xyx.$

$$(xyx)^{2t}(yx)(xyx)^{3t} = (xyx)^{2t-1}(xyx)(yx)(xyx)^{3t} =$$
$$(xyx)^{2t-1}(xy)(xyx)(xyx)^{3t} = (xyx)^{2t-1}(xy)(xyx)^{3t+1}$$

Then we put

$$\pi(w) = \pi(w_1) \circ u^{\alpha_1} \circ \pi(w_2) \circ u^{\alpha_2} \circ \cdots \circ \pi(w_m) \circ u^{\alpha_m} \circ \pi(w_{m+1}),$$

with tuple $(\alpha_1, \ldots, \alpha_n)$ maximal w.r. to left lexicographical order.

Theorem. (Kharlampovich, Miasnikov)

Each finitely generated fully residually free group can be embedded into F_n with suitable n and $u_1, \ldots, u_n$.

So the results about f.g. subgroups of $F^{\mathbb{Z}[t]}$ also hold for f.g. fully residually free groups.

Graphs labelled by infinite words

Let G be a finitely generated subgroup of F_n.

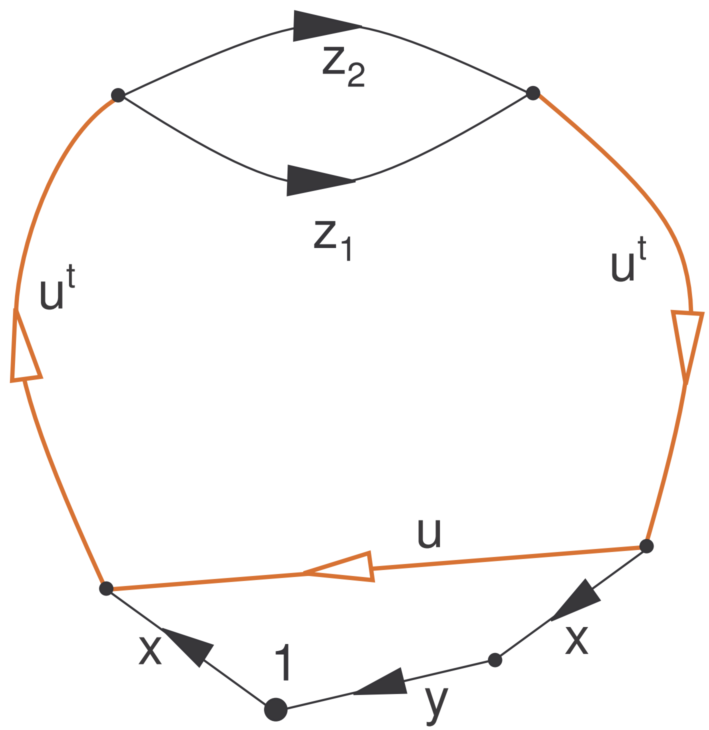
Denote u's used to obtain F_n by $u_1, \ldots, u_n$.

Let Γ be a combinatorial directed graph with edges labelled by elements of X and exponents of $u_1, \ldots, u_n$.

Choose a base vertex v, and consider group

$$L(\Gamma, v) = \{\mu(p) | p \text{ a loop at } v\}$$

Example



Here, $u = xyx$, and the group is
$G = L(\Gamma, 1) = \langle xu^t z_1 u^t xy, xu^t z_2 u^t xy \rangle$.

Definition

If a labelled directed graph Γ_G is such that

> there exists a path p with $o(p) = v_1, t(p) = v_2$ and $\mu(p) = g$
>
> iff
>
> there exists a path p' with $o(p) = v_1, t(p) = v_2$ and
>
> $\mu(p) = \pi(g)$,

then the graph is called U-folded.

One can also give an explicit combinatorial definition of a U-folded graph, which allows to construct U-folded graphs effectively.

Theorem. (Miasnikov, Remeslennikov, Serbin)

Let G be a finitely generated subgroup of $F^{\mathbb{Z}[t]}$. Then there exists a finite U-folded labeled directed graph Γ_G with $L(\Gamma_G) = G$.

Moreover, Γ_G can be constructed effectively, given generators of G.

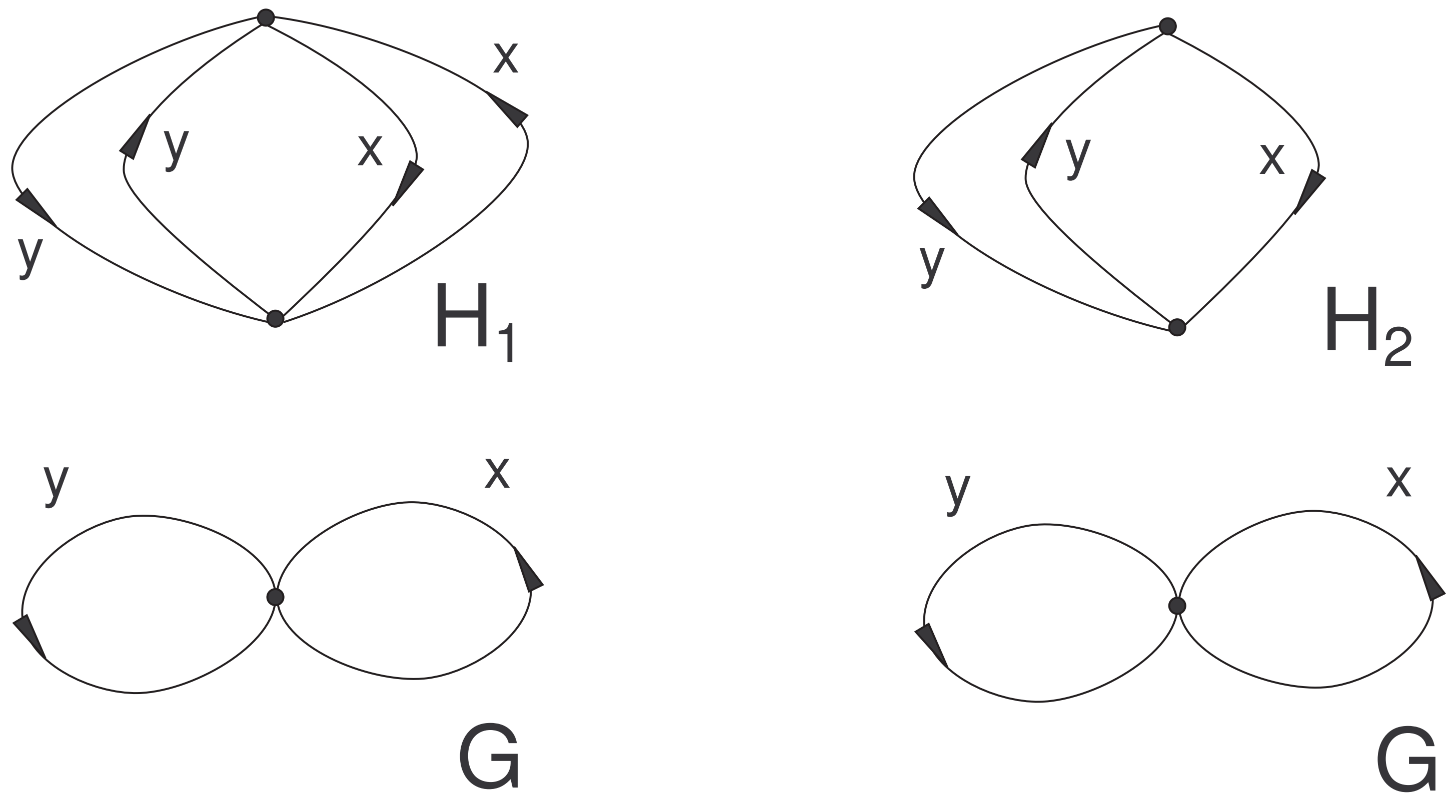
Kharlampovich, Miasnikov, Remeslennikov, Serbin showed that U-folded graphs can be employed to solve

- Subgroup Membership Problem,
- Intersection Problem,
- Conjugacy Problem,
- other algorithmic problems

Now we use this technique to solve Finite Index Problem.

Finite index conditions

Stallings graphs provide a simple way to decide if a subgroup H of a free group G is of finite index. The algorithm was basically to check if label of any loop in the Stallings graph for G could be read as label of a path in the Stallings graph for H.



$$G = \langle x, y \rangle$$

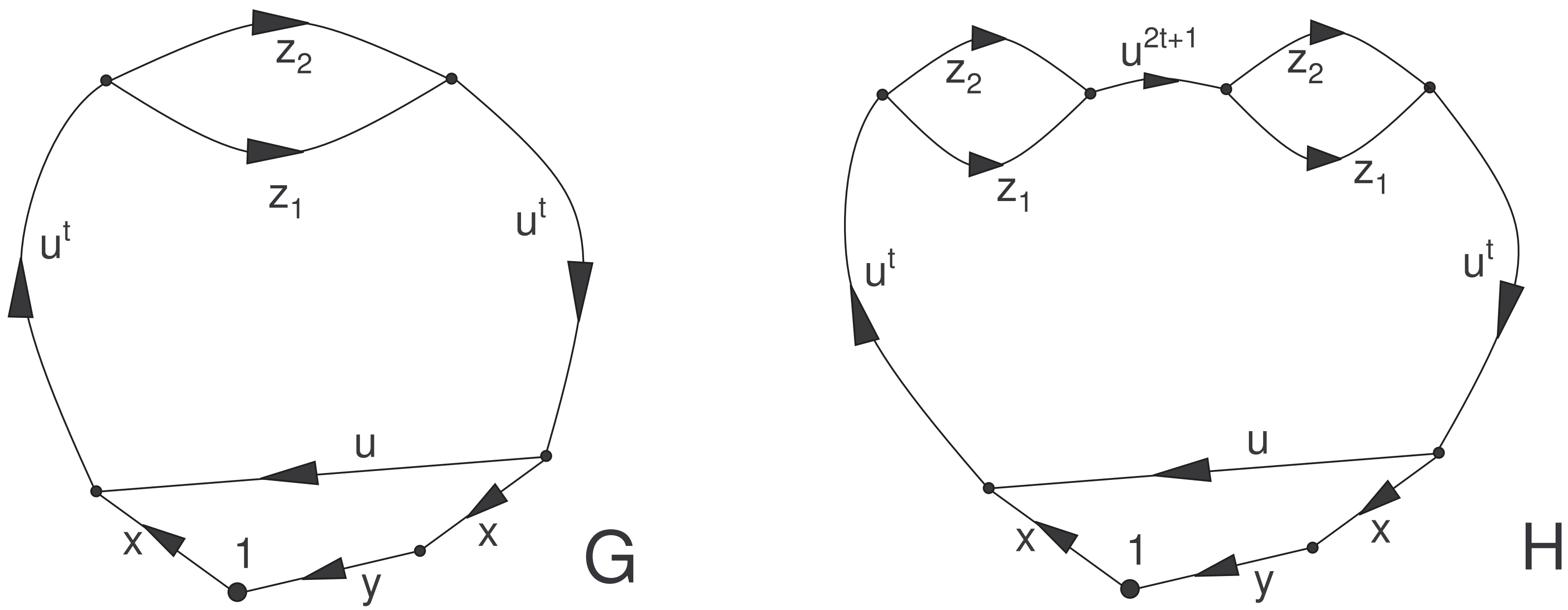
$$H_1 = \langle x^2, y^2, xy \rangle,\ |G : H_1| = 2 < \infty$$

$$H_2 = \langle yx, y^{-1}x \rangle,\ |G : H_2| = \infty$$

Question

Is there a similar test for finitely generated subgroup of $F^{\mathbb{Z}[t]}$ given by U-folded graphs?

Straightforward translation of the algorithm to U-graphs does not work:



$u = xyx$, $G = \langle a = xu^t z_1 u^t xy, b = xu^t z_2 u^t xy \rangle$, $H = \langle a^2, b^2, ab \rangle$, $|G : H| = 2 < \infty$

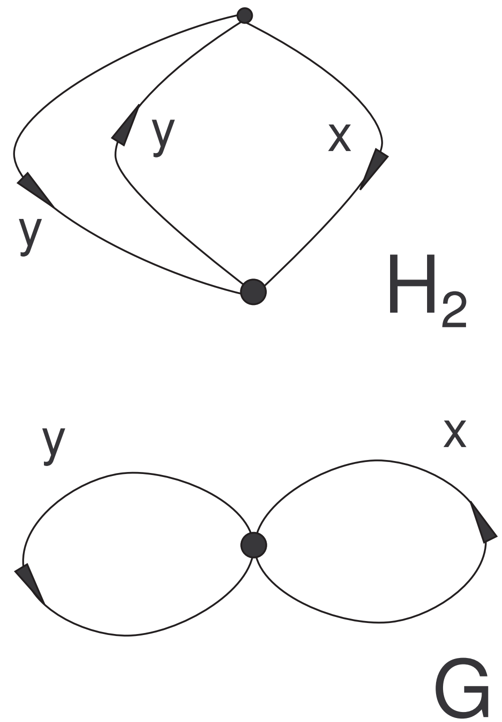
In what follows, we denote a limit group by G and its subgroup by H. We assume they are given by U-folded graphs Γ and Δ respectively.

Observation

To conclude that index is infinite, it is not enough to find a loop in Γ, whose label cannot be read in Δ.

It also matters if one can or cannot produce infinite amount of cosets once such a loop is found.

In case when G is free and is Γ is a bouquet of circles, that was automatic.



We cannot read x in H_2, so we cannot read any $xw \in G$, w a word that does not start with x^{-1}.

Since in a U-folded graph reading a label is equivalent to reading its normal form, it is enough to examine paths whose labels are normal forms, or S-paths.

Theorem

Let G be a f.g. subgroup of $F^{\mathbb{Z}[t]}$ and f.g. $H \leq G$. Then the following are equivalent:

(1) $|G : H| < \infty$,

(2) there exists a finite U-folded graph Δ with a vertex v such that $H = L(\Delta, v)$ and for every $g \in G$ there exists a path p in Δ such that $o(p) = v,\ \mu(p) = \pi(g)$.

However, this theorem depends on a particular graph Δ.

To make the test work effectively given arbitrary graph representing Γ, we investigate infinite S-paths.

Theorem

Given $G \geq H$ defined by Γ and Δ, the following statements are equivalent:

(1) $[G : H] < \infty$,

(3) for each infinite S-path in Γ, its label is also readable in Δ.

Idea behind the theorem

Suppose there is a finite S-path p in Γ that is not doubled in Δ. Whether we can use it to produce infinitely many cosets, depends on whether or not we can extend it into an infinite S-path.

If we can, index is infinite;

if we cannot extend any non-doubled finite S-path, then all possible non-doubled pieces occur in the "ends" of these paths. It is possible to put a bound on how long the "end" can be, explaining why we can cover all G with a finite number of cosets.

This theorem doesn't provide an effective criterion. To make the theorem effective, we need to restrict the test to checking a finite number of paths.

Definition

An infinite S-path p is called periodic if $p = p_1(p_2)^\infty$.

One can prove an analogue of Pumping Lemma stating that doubling all infinite S-paths is equivalent to doubling all periodic S-paths of a bounded “content”.

Content is, roughly speaking, number of symbols in $\mu(p_1) + \mu(p_2)$.

Theorem

Given $G \geq H$ defined by U-folded graphs Γ and Δ, one can compute a number N such that the following statements are equivalent:

(1) $[G : H] < \infty$,

(3) for each infinite S-path (in particular, each periodic S-path) in Γ, its label is also readable in Δ,

(4) for each periodic S-path of content $\leq N$ in Γ, its label is also readable in Δ.

One can show that condition (4) can be checked effectively.

Theorem

Suppose $H \leq G \leq F^{\mathbb{Z}[t]}$, G, H f.g. Then there is an algorithm deciding whether $[G : H] < \infty$.

These results have a number of corollaries.

Corollary 1 (Greenberg-Stallings Theorem)

Let G_1, G_2 be finitely generated subgroups of $F^{\mathbb{Z}[t]}$. If $H \leq G_1 \cap G_2$ is finitely generated and $|G_1 : H| < \infty$, $|G_2 : H| < \infty$ then $|\langle G_1, G_2 \rangle : H| < \infty$.

For example, $H = G_1 \cap G_2$ is finitely generated if G_1, G_2 are, as shown by Kharlampovich, Myasnikov, Remeslennikov, and Serbin (2004).

Commensurator

Let H be a subgroup of a group G. The commensurator $\mathrm{Comm}_G(H)$ of H in G is defined as

$$\mathrm{Comm}_G(H) = \{g \in G \mid |H : H \cap gHg^{-1}| < \infty \text{ and } |gHg^{-1} : H \cap gHg^{-1}| < \infty$$

Combining decidability of finite index with results of Kharlampovich, Myasnikov, Remeslennikov and Serbin (2004), we immediately obtain the following

Corollary 2

Let $H \leq G$ be two noncommutative f.g. subgroups of $F^{\mathbb{Z}[t]}$. Then $\mathrm{Comm}_G(H)$ is finitely generated, and its generating set can be found effectively.

Analogue of Greenberg-Stallings Theorem allows to prove

Corollary 3

Let $H \leq G$ be two noncommutative f.g. subgroups of $F^{\mathbb{Z}[t]}$. Then $|\mathrm{Comm}_G(H) : H| < \infty$.

Remark

This theorem might not hold if H is commutative. Obvious example is $G = \langle u, u^t \rangle$, $H = \langle u \rangle$. One can add an x to G to make it noncommutative: $G = \langle x, u, u^t \rangle$, $H = \langle u \rangle$.

Corollary 4 (Schreier-like bound)

Let G be a f.g. subgroup of $F^{\mathbb{Z}[t]}$ and let $H \leq G$ be its f.g. non-abelian subgroup. Then there exists a natural number $N(H)$ such that for every $K \leq G$ containing H, if $|K : H| < \infty$ then $|K : H| < N(H)$.