Membership problem in $\mathbb{Z}^n$-free groups

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Free groups:

$F(X) \hookrightarrow R(\mathbb{Z}, X)$.

Graphs labeled by X, Stallings foldings (83).

Solution to numerous algorithmic problems.

L. Marcus-Epstein (2005).
Free products and amalgams of finite groups.

Fully residually free groups:

If G is a f.g. fully residually free group, then $G \hookrightarrow F_n$, where

$$F(X) = F_0 < F_1 < \ldots < F_n,$$

where $F_k = \langle F_{k-1}, u_k^\alpha | \alpha \in \mathbb{Z}[t] \rangle$ (Miasnikov, Kharlampovich).

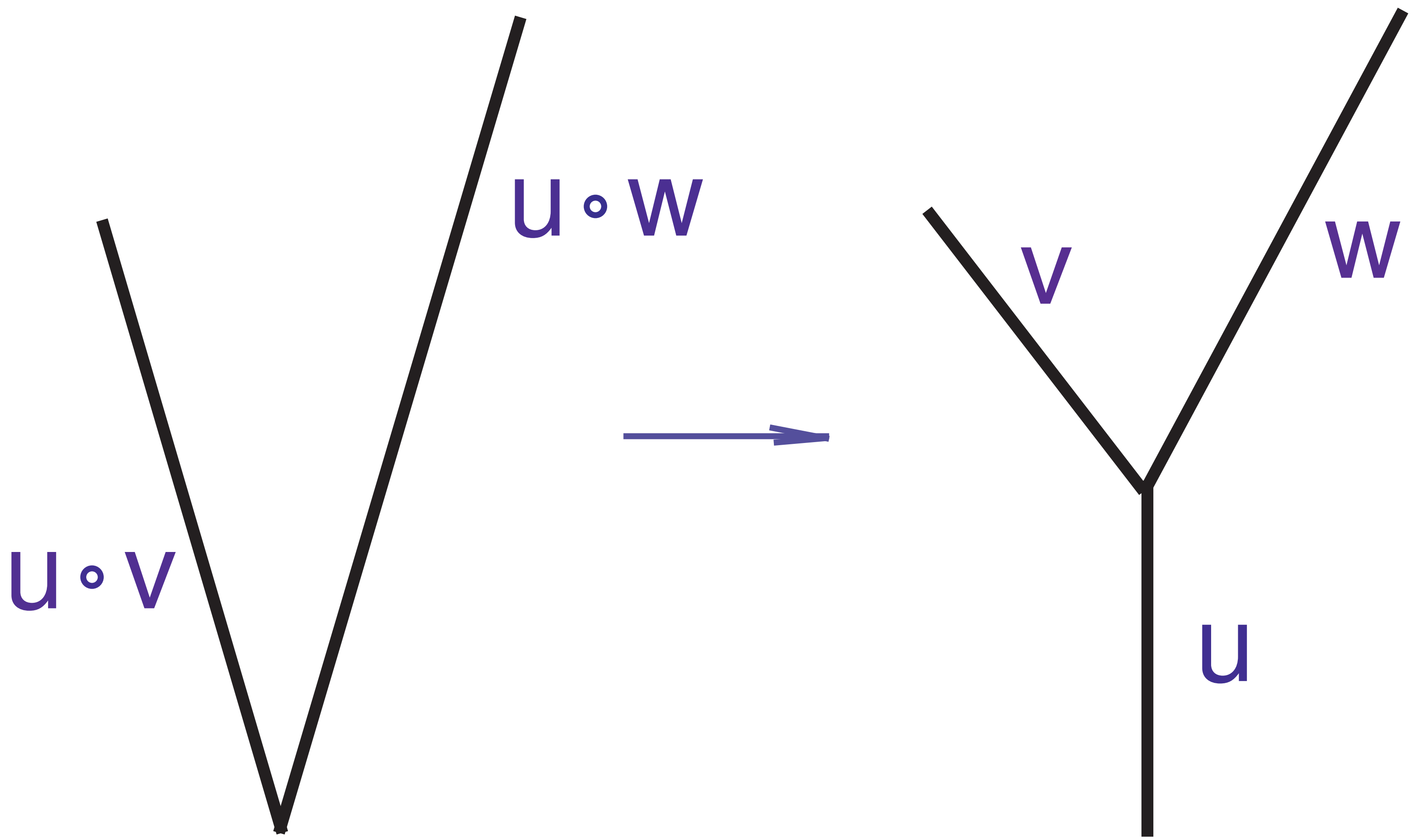
Moreover, $G \in R^*(\mathbb{Z}^N, X)$ (Miasnikov, Remeslennikov, Serbin).

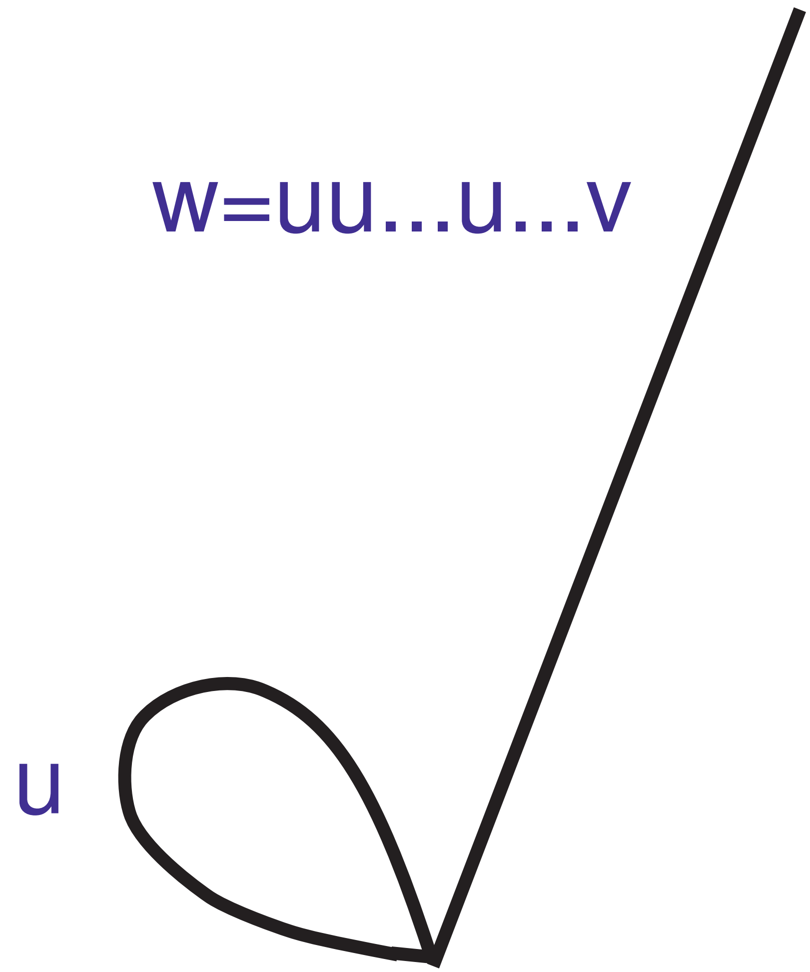
Graphs labeled by X and u_i^α. U-foldings.

Kharlampovich, Miasnikov, Remeslennkov, Serbin, Nikolaev (2004–2008).
Solutions to Subgroup Membership, Intersection, Conjugacy, Normality, Malnormality, Finite Index and other problems.

Proof of Subgroup Separability in terms of U-graphs is not known.

Can we organize Stallings-like graph technique for arbitrary f.g. subgroups of $R^*(\mathbb{Z}^n, X)$?


u ∘ w
u ∘ v
v
w
u


w=uu...u...v
u

$\mathbb{Z}^n$-trees

Let Λ be an ordered abelian group, for example $\mathbb{Z}^n$ with right lexicographic order.

The following definition is due to Morgan and Shalen (1984).

A Λ-tree is a geodesic Λ-metric space (X, d) such that for all $x, y, z \in X$
$[x, y] \cap [y, z] = [x, w]$ for some $w \in X$,
$[x, y] \cap [y, z] = \{y\} \Rightarrow [x, z] = [x, y] \cup [y, z]$.

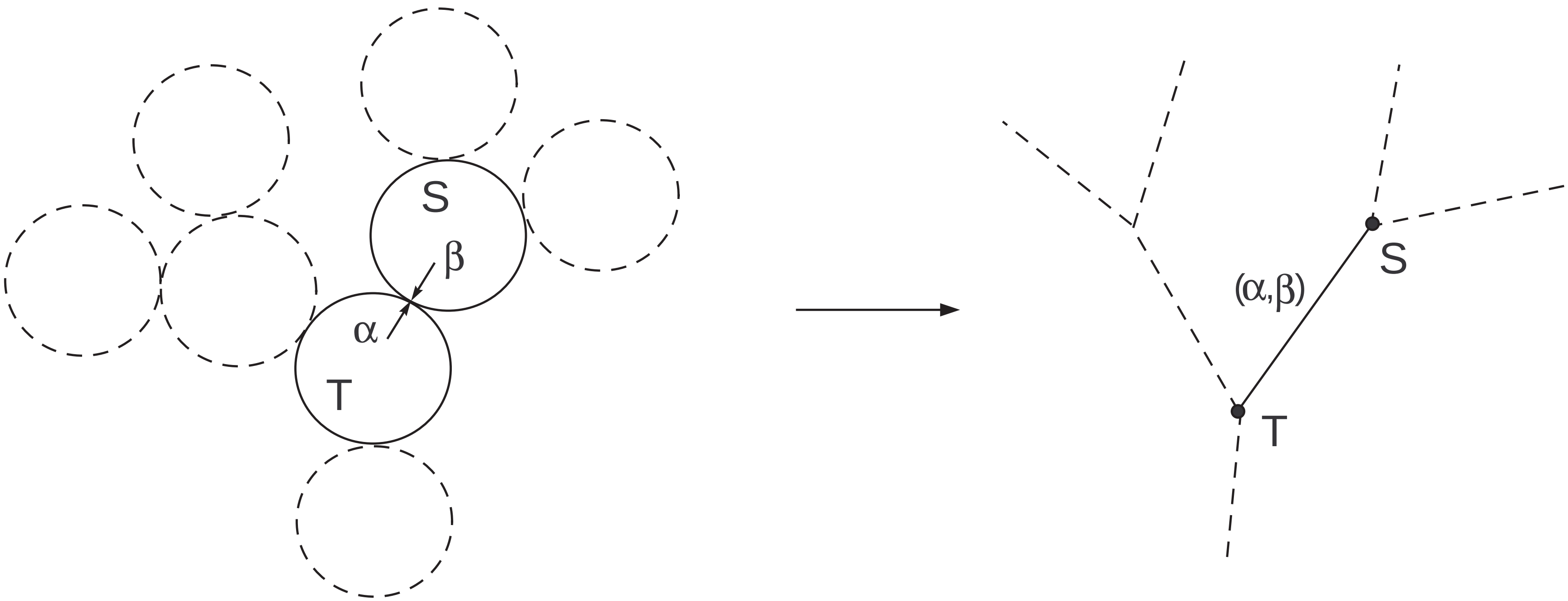
Examples.

$n = 1$.

$\mathbb{Z}$-tree is a "usual" simplicial tree.

$n = 2$.

$\mathbb{Z}^2$-tree can be viewed as a "tree of $\mathbb{Z}$-trees".



An isometric action of a group on a Λ-tree X is free if there are no inversions and the stabilizer of each point of X is trivial. We say that a group G is Λ-free if G admits such an action on some Λ-tree.

$\mathbb{Z}^n$-free groups

The following is due to Myasnikov–Remeslennikov–Serbin and Chiswell.

Theorem. Let G be a finitely generated group. Then the following are equivalent:

- there exists an embedding $G \hookrightarrow R^*(\mathbb{Z}^n, X)$,
- G has a free Lyndon length function with values in $\mathbb{Z}^n$,
- G acts freely on $\mathbb{Z}^n$-tree.

Martino, O Rourke (2004), Guirardel (2004).

1. (MR) $\mathbb{Z}^n$-free groups are commutation transitive, and any abelian subgroup of a $\mathbb{Z}^n$-free group is free abelian of rank at most n.
2. (MR) $\mathbb{Z}^n$-free groups are coherent.
3. (G) $\mathbb{Z}^n$-free groups are hyperbolic relative to maximal abelian subgroups.
4. (MR) A finitely generated $\mathbb{Z}^n$-free group all of whose maximal abelian subgroups are cyclic is word hyperbolic (as are all its finitely generated subgroups).
5. (MR) Word Problem is decidable in any $\mathbb{Z}^n$-free group.
6. (MR) Class of Z^n-free (for some n) groups is closed under amalgamated products along maximal abelian subgroups.
7. (Wise) Finitely generated Z^n-free groups admit hierarchy in Dani Wise's sense.

Theorem.

(Kharlampovich–Myasnikov–Remeslennikov–Serbin)
Finitely generated G has a regular free action on a Z^n-tree if and only if G can be represented as a union of a finite series of groups

$$G_1 < G_2 < \ldots < G_n = G,$$

where

1. G_i has a regular free action on a Z^i-tree (that is, G_1 is a free group),
2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian and length-isomorphic.

Good intentions.

Given finitely generated $\mathbb{Z}^n$-free group G,

1. based on structure theorem and Britton's lemma, define normal form of elements of G,
2. build (not folded) graph labeled by infinite words that recognizes G,
3. "fold" it so that it accepts normal forms of elements of G,
4. enjoy solving algorithmic problems.

Good intention $\#1$ fails.

Normal forms similar to ones in limit groups are unreasonably technically complicated. Instead of a unique normal form, for each $g \in G$ we define an infinite set of words $\Pi(g)$.

Denote in last theorem

$$G_n = \langle G_{n-1}, T_{n-1} \mid w^{-1} C_w w \stackrel{\phi_w}{=} D_w,\ w \in T_{n-1},\ \rangle.$$

As an infinite word, element w starts with "positive" infinite power of any element of C_w and ends with "positive" infinite power of any element of D_w.

Example:

$$C_w = \langle xy \rangle,\ D_w = \langle zx \rangle,$$

$$w = xyxy \cdots zxzx,$$

$$(x^{-1}z^{-1}x^{-1}z^{-1} \dots y^{-1}x^{-1}y^{-1}x^{-1})xy(xyxy \cdots zxzx) = zx.$$

Define finite alphabet $\mathcal{B}(G)$ to be union $X \cup T_1 \cup \ldots \cup T_{n-1}$.

Building folded $\mathcal{B}(G)$-graph.

Primary (short-term) goal:
build $\mathcal{B}(G)$-graph that can be used to solve subgroup membership problem.

Long-term goal:
build $\mathcal{B}(G)$-graph that can be reasonably used to solve other algorithmic problems.

Theorem. (Nikolaev–Serbin) For a fixed G, there exists algorithm that, given a $\mathcal{B}(G)$-graph Γ produces Γ', that recognizes the same group, with the following property:
if there exists path p in Γ' such that

$$o(p) = v_1, e(p) = v_2, \mu(p) = g,$$

then there exists path q such that

$$o(q) = v_1, e(q) = v_2, \mu(p) \in \Pi(g).$$

The latter for a given g can be checked effectively.

Good: Solved uniform subgroup membership problem (and power problem).

Bad: Solution to other algorithmic problems (even intersection problem) will be rather involved.