Knapsack problems in products of groups

Andrey Nikolaev
(Stevens Institute of Technology)

CCNY October 24, 2014

Based on joint work with E.Frenkel and A.Ushakov

Basic idea:

Take a classic algorithmic problem from computer science (traveling salesman, Post correspondence, knapsack, ...) and translate it into group-theoretic setting.

Example: Post correspondence problem

Let A be an alphabet, $|A| \geq 2$.

The classic Post correspondence problem (PCP)

Given a finite set of pairs $(g_1, h_1), \ldots, (g_k, h_k)$ of elements of A^* determine if there is a non-empty word $w(x_1, \ldots, x_k) \in X^*$ such that $w(g_1, \ldots, g_k) = w(h_1, \ldots, h_k)$ in A^*.

Example: Post correspondence problem

Matching dominoes: top = bottom

g_{i_1}	g_{i_2}	g_{i_3}	$\dots$	g_{i_n}
h_{i_1}	h_{i_2}	h_{i_3}	$\dots$	h_{i_n}

Decidable if number of pairs is $k \leq 3$. Undecidable if $k \geq 7$. Unknown if $4 \leq k \leq 6$.

Example: Post correspondence problem

Matching dominoes: top = bottom

g_{i_1}	g_{i_2}	g_{i_3}	$\ldots$	g_{i_n}
h_{i_1}	h_{i_2}	h_{i_3}	$\ldots$	h_{i_n}

Decidable if number of pairs is $k \leq 3$. Undecidable if $k \geq 7$. Unknown if $4 \leq k \leq 6$.

PCP in groups

Translating **PCP** to groups:

$A^* \rightsquigarrow$ f.g. group G,
words $g_i, h_i \rightsquigarrow$ group elements g_i, h_i given as words in generators,
word $w \rightsquigarrow$ group word,
right?

The above is trivial:
(a) $w = xx^{-1}$. Only allow non-trivial reduced words.
(b) G abelian, $w = [x, y]$. Only allow words that are not identities of G.

Translating **PCP** to groups:

$A^* \rightsquigarrow$ f.g. group G,
words $g_i, h_i \rightsquigarrow$ group elements g_i, h_i given as words in generators,
word $w \rightsquigarrow$ group word,
right?

The above is trivial:
(a) $w = xx^{-1}$. Only allow non-trivial reduced words.
(b) G abelian, $w = [x, y]$. Only allow words that are not identities of G.

Translating **PCP** to groups:

$A^* \rightsquigarrow$ f.g. group G,
words $g_i, h_i \rightsquigarrow$ group elements g_i, h_i given as words in generators,
word $w \rightsquigarrow$ group word,
right?

The above is trivial:
(a) $w = xx^{-1}$. Only allow non-trivial reduced words.
(b) G abelian, $w = [x, y]$. Only allow words that are not identities of G.

Translating **PCP** to groups:

$A^* \rightsquigarrow$ f.g. group G,
words $g_i, h_i \rightsquigarrow$ group elements g_i, h_i given as words in generators,
word $w \rightsquigarrow$ group word,
right?

The above is trivial:
(a) $w = xx^{-1}$. Only allow non-trivial reduced words.
(b) G abelian, $w = [x, y]$. Only allow words that are not identities of G.

Example: Post correspondence problem

Variations of **PCP** in groups turn out to be closely related to:

- double-endo-twisted conjugacy problem
  (find $w \in G$ s.t. $uw^{\varphi} = w^{\psi} v$),
- equalizer problem
  (find the subgroup of elements g s.t. $\varphi(g) = \psi(g)$),
- hereditary word problem
  (word problem in any quotient of G by a subgroup f.g. as a normal subgroup).

Example: Post correspondence problem

Variations of **PCP** in groups turn out to be closely related to:

- double-endo-twisted conjugacy problem
  (find $w \in G$ s.t. $uw^{\varphi} = w^{\psi}v$),
- equalizer problem
  (find the subgroup of elements g s.t. $\varphi(g) = \psi(g)$),
- hereditary word problem
  (word problem in any quotient of G by a subgroup f.g. as a normal subgroup).

Example: Post correspondence problem

Variations of **PCP** in groups turn out to be closely related to:

- double-endo-twisted conjugacy problem (find $w \in G$ s.t. $uw^{\varphi} = w^{\psi}v$),
- equalizer problem (find the subgroup of elements g s.t. $\varphi(g) = \psi(g)$),
- hereditary word problem (word problem in any quotient of G by a subgroup f.g. as a normal subgroup).

Example: Post correspondence problem

Variations of **PCP** in groups turn out to be closely related to:

- double-endo-twisted conjugacy problem
  (find $w \in G$ s.t. $uw^{\varphi} = w^{\psi}v$),
- equalizer problem
  (find the subgroup of elements g s.t. $\varphi(g) = \psi(g)$),
- hereditary word problem
  (word problem in any quotient of G by a subgroup f.g. as a normal subgroup).

Example: Post correspondence problem

Variations of **PCP** in groups turn out to be closely related to:

- double-endo-twisted conjugacy problem
  (find $w \in G$ s.t. $uw^{\varphi} = w^{\psi}v$),
- equalizer problem
  (find the subgroup of elements g s.t. $\varphi(g) = \psi(g)$),
- hereditary word problem
  (word problem in any quotient of G by a subgroup f.g. as a normal subgroup).

Example: Post correspondence problem

Variations of **PCP** in groups turn out to be closely related to:

- double-endo-twisted conjugacy problem
  (find $w \in G$ s.t. $uw^{\varphi} = w^{\psi} v$),
- equalizer problem
  (find the subgroup of elements g s.t. $\varphi(g) = \psi(g)$),
- hereditary word problem
  (word problem in any quotient of G by a subgroup f.g. as a normal subgroup).

The classic subset sum problem (SSP):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

$$\varepsilon_1 a_1 + \ldots + \varepsilon_k a_k = a$$

for some $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$.

SSP for a group G:

Given $g_1, \ldots, g_k, g \in G$ decide if

$$g_1^{\varepsilon_1} \ldots g_k^{\varepsilon_k} = g$$

for some $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$.

Elements in G are given as words in a fixed set of generators of G.

The classic subset sum problem (SSP):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

$$\varepsilon_1 a_1 + \ldots + \varepsilon_k a_k = a$$

for some $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$.

SSP for a group G:

Given $g_1, \ldots, g_k, g \in G$ decide if

$$g_1^{\varepsilon_1} \ldots g_k^{\varepsilon_k} = g$$

for some $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$.

Elements in G are given as words in a fixed set of generators of G.

The classic subset sum problem (SSP):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

$$\varepsilon_1 a_1 + \ldots + \varepsilon_k a_k = a$$

for some $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$.

SSP for a group G:

Given $g_1, \ldots, g_k, g \in G$ decide if

$$g_1^{\varepsilon_1} \ldots g_k^{\varepsilon_k} = g$$

for some $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$.

Elements in G are given as words in a fixed set of generators of G.

Algorithmic set-up

Classic SSP is pseudopolynomial

- If input is given in unary, **SSP** is in **P**,
- if input is given in binary, **SSP** is **NP**-complete.

The complexity of **SSP**(G) does not depend on a finite generating set, but may depend on a generating set if infinite ones are allowed.

For example:

SSP$(\mathbb{Z})$

- **SSP**$(\mathbb{Z}) \in$ **P** if $\mathbb{Z}$ is generated by $\{1\}$,
- **SSP**$(\mathbb{Z})$ is **NP**-complete if $\mathbb{Z}$ is generated by $\{2^n \mid n \in \mathbb{N}\}$.

Algorithmic set-up

Classic SSP is pseudopolynomial

- If input is given in unary, **SSP** is in **P**,
- if input is given in binary, **SSP** is **NP**-complete.

The complexity of **SSP**(G) does not depend on a finite generating set, but may depend on a generating set if infinite ones are allowed.

For example:

SSP$(\mathbb{Z})$

- **SSP**$(\mathbb{Z}) \in$ **P** if $\mathbb{Z}$ is generated by $\{1\}$,
- **SSP**$(\mathbb{Z})$ is **NP**-complete if $\mathbb{Z}$ is generated by $\{2^n \mid n \in \mathbb{N}\}$.

Algorithmic set-up

Classic SSP is pseudopolynomial

- If input is given in unary, **SSP** is in **P**,
- if input is given in binary, **SSP** is **NP**-complete.

The complexity of **SSP**(G) does not depend on a finite generating set, but may depend on a generating set if infinite ones are allowed.

For example:

SSP$(\mathbb{Z})$

- **SSP**$(\mathbb{Z}) \in$ **P** if $\mathbb{Z}$ is generated by $\{1\}$,
- **SSP**$(\mathbb{Z})$ is **NP**-complete if $\mathbb{Z}$ is generated by $\{2^n \mid n \in \mathbb{N}\}$.

Complexity of **SSP**(G):

Group	Complexity	Why
Nilpotent	**P**	Poly growth
$\mathbb{Z} \wr \mathbb{Z}$	**NP**-complete	$\mathbb{Z}^\omega$, **ZOE**
Free metabelian	**NP**-complete	$\mathbb{Z} \wr \mathbb{Z}$
Thompson's F	**NP**-complete	$\mathbb{Z} \wr \mathbb{Z}$
$BS(1, p)$	**NP**-complete	Binary **SSP**$(\mathbb{Z})$
Hyperbolic	**P**	Log depth

Note that the **NP**-completeness is despite unary input.

Complexity of **SSP**(G):

Group	Complexity	Why
Nilpotent	**P**	Poly growth
$\mathbb{Z} \wr \mathbb{Z}$	**NP**-complete	$\mathbb{Z}^\omega$, **ZOE**
Free metabelian	**NP**-complete	$\mathbb{Z} \wr \mathbb{Z}$
Thompson's F	**NP**-complete	$\mathbb{Z} \wr \mathbb{Z}$
$BS(1, p)$	**NP**-complete	Binary **SSP**$(\mathbb{Z})$
Hyperbolic	**P**	Log depth

Note that the **NP**-completeness is despite unary input.

Knapsack problems in groups

Three principle Knapsack type (decision) problems in groups:

- **SSP** subset sum,
- **KP** knapsack,
- **SMP** submonoid membership.

The knapsack problem in groups

The classic knapsack problem (KP):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

$$n_1 a_1 + \ldots + n_k a_k = a$$

for some non-negative integers $n_1, \ldots, n_k$.

The knapsack problem (KP) for G:

Given $g_1, \ldots, g_k, g \in G$ decide if

$$g_1^{n_1} \ldots g_k^{n_k} = g$$

for some non-negative integers $n_1, \ldots, n_k$.

There are minor variations of this problem, for instance, integer **KP**, when n_i are arbitrary integers. They are all similar, we omit them here.

The classic knapsack problem (KP):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

$$n_1 a_1 + \ldots + n_k a_k = a$$

for some non-negative integers $n_1, \ldots, n_k$.

The knapsack problem (KP) for G:

Given $g_1, \ldots, g_k, g \in G$ decide if

$$g_1^{n_1} \ldots g_k^{n_k} = g$$

for some non-negative integers $n_1, \ldots, n_k$.

There are minor variations of this problem, for instance, integer **KP**, when n_i are arbitrary integers. They are all similar, we omit them here.

The knapsack problem in groups

The classic knapsack problem (KP):

Given $a_1, \ldots, a_k, a \in \mathbb{Z}$ decide if

$$n_1 a_1 + \ldots + n_k a_k = a$$

for some non-negative integers $n_1, \ldots, n_k$.

The knapsack problem (KP) for G:

Given $g_1, \ldots, g_k, g \in G$ decide if

$$g_1^{n_1} \ldots g_k^{n_k} = g$$

for some non-negative integers $n_1, \ldots, n_k$.

There are minor variations of this problem, for instance, integer **KP**, when n_i are arbitrary integers. They are all similar, we omit them here.

The knapsack problem in groups

The knapsack problem in groups is closely related to the big powers method, which appeared long before any complexity considerations.

The submonoid membership problem in groups

Submonoid membership problem (SMP):

Given a finite set $A = \{g_1, \ldots, g_k, g\}$ of elements of G decide if g belongs to the submonoid generated by A, i.e., if $g = g_{i_1}, \ldots, g_{i_s}$ for some $g_{i_j} \in A$.

If the set A is closed under inversion then we have the subgroup membership problem in G.

Bounded variations

It makes sense to consider the bounded versions of **KP** and **SMP**, they are always decidable in groups with decidable word problem.

The bounded knapsack problem (BKP) for G:

decide, when given $g_1, \ldots, g_k, g \in G$ and $1^m \in \mathbb{N}$, if $g =_G g_1^{\varepsilon_1} \ldots g_k^{\varepsilon_k}$ for some $\varepsilon_i \in \{0, 1, \ldots, m\}$.

BKP is **P**-time equivalent to **SSP** in G.

Bounded variations

It makes sense to consider the bounded versions of **KP** and **SMP**, they are always decidable in groups with decidable word problem.

The bounded knapsack problem (BKP) for G:

decide, when given $g_1, \ldots, g_k, g \in G$ and $1^m \in \mathbb{N}$, if $g =_G g_1^{\varepsilon_1} \ldots g_k^{\varepsilon_k}$ for some $\varepsilon_i \in \{0, 1, \ldots, m\}$.

BKP is **P**-time equivalent to **SSP** in G.

Bounded variations

Bounded submonoid membership problem (BSMP) for G:

Given $g_1, \ldots g_k, g \in G$ and $1^m \in \mathbb{N}$ (in unary) decide if g is equal in G to a product of the form $g = g_{i_1} \cdots g_{i_s}$, where $g_{i_1}, \ldots, g_{i_s} \in \{g_1, \ldots, g_k\}$ and $s \leq m$.

Known results [MNU]

SSP and **BKP**:

- **NP**-complete in $\mathbb{Z} \wr \mathbb{Z}$, free metabelian, Thompson's F, $BS(m, n), m \neq \pm n$.
- **P**-time in f.g. v. nilpotent groups, hyperbolic groups, $BS(n, \pm n)$.

BSMP:

- **NP**-complete in $F_2 \times F_2$ (therefore **NP**-hard in any group that contains $F_2 \times F_2$, e.g. $B_{\geq 5}$, $GL(\geq 4, \mathbb{Z})$, partially commutative with induced $\boxtimes$.)
- **P**-time in f.g. v. nilpotent groups, hyperbolic groups.

KP:

- **P**-time in abelian groups, hyperbolic groups.

SSP vs group-theoretic constructions

What about group-theoretic constructions?

Q1 Does **SSP** carry from G, H to $G * H$?

A1 That's not the right question.

Q2 Does **SSP** in $G \times H$ behave like the word problem or like the membership problem?

A2 Both!

SSP vs group-theoretic constructions

What about group-theoretic constructions?

Q1 Does **SSP** carry from G, H to $G * H$?

A1 That's not the right question.

Q2 Does **SSP** in $G \times H$ behave like the word problem or like the membership problem?

A2 Both!

SSP vs group-theoretic constructions

What about group-theoretic constructions?

Q1 Does **SSP** carry from G, H to $G * H$?

A1 That's not the right question.

Q2 Does **SSP** in $G \times H$ behave like the word problem or like the membership problem?

A2 Both!

SSP vs group-theoretic constructions

What about group-theoretic constructions?

Q1 Does **SSP** carry from G, H to $G * H$?

A1 That's not the right question.

Q2 Does **SSP** in $G \times H$ behave like the word problem or like the membership problem?

A2 Both!

SSP vs group-theoretic constructions

What about group-theoretic constructions?

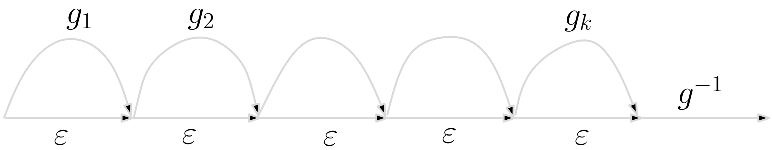
Q1 Does **SSP** carry from G, H to $G * H$?

A1 That's not the right question.

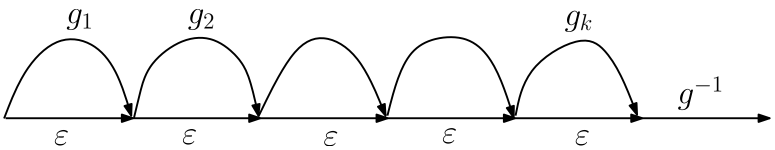
Q2 Does **SSP** in $G \times H$ behave like the word problem or like the membership problem?

A2 Both!

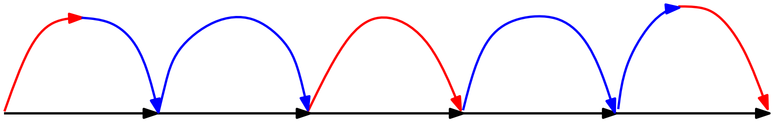
What an instance of **SSP**(G) looks like?



What an instance of **SSP**(G) looks like?

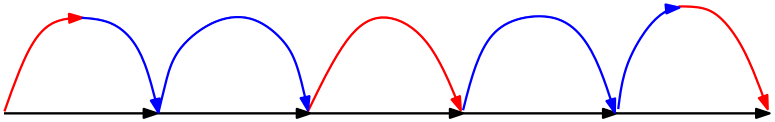


Consider **SSP**($G * H$)



Try to solve it using **SSP**(G) and **SSP**(H).

Consider **SSP**$(G * H)$

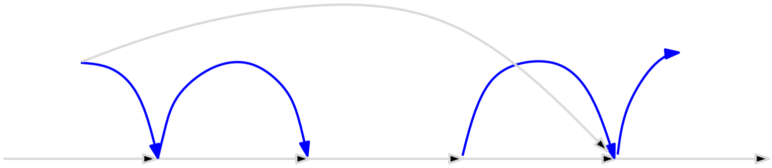


Try to solve it using **SSP**(G) and **SSP**(H).

SSP and free products

Look at the G part:

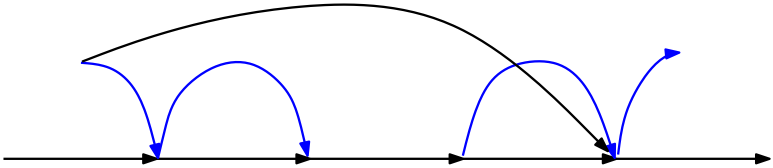
Solve all occurring instances of **SSP**(G):



SSP and free products

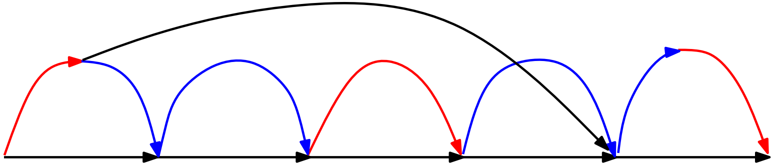
Look at the G part:

Solve all occurring instances of **SSP**(G):

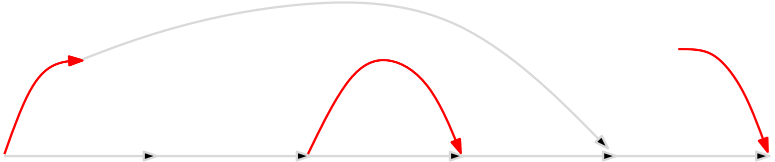


SSP and free products

Bring back H part:



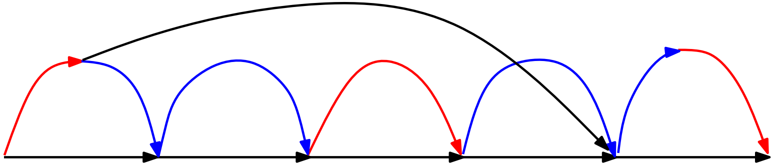
Look at H part separately:



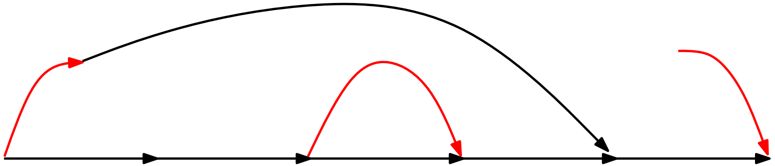
This is not **SSP** anymore! ($3 \neq 2^m$ choices of paths.)

SSP and free products

Bring back H part:

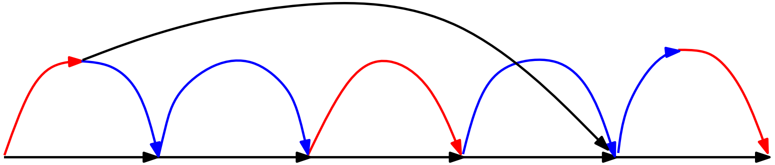


Look at H part separately:

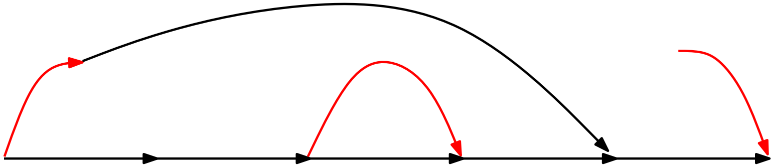


This is not **SSP** anymore! ($3 \neq 2^m$ choices of paths.)

Bring back *H* part:



Look at *H* part separately:



This is not **SSP** anymore! ($3 \neq 2^m$ choices of paths.)

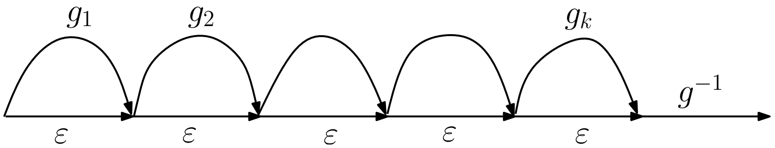
In this context, it is natural to consider so-called Acyclic Graph Problem:

The acyclic graph problem AGP(G, X)

Given an acyclic directed graph Γ labeled by letters in $X \cup X^{-1} \cup \{\varepsilon\}$ with two marked vertices, α and ω, decide whether there is an oriented path in Γ from α to ω labeled by a word w such that $w = 1$ in G.

AGP(G)

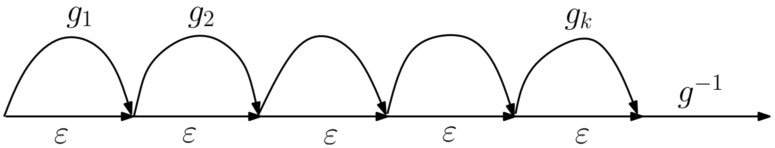
AGP(G) generalizes **SSP**(G) (i.e. **SSP**(G) is **P**-time reducible to **AGP**(G)):



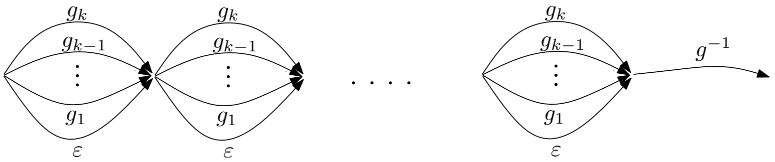
AGP(G) generalizes **BSMP**(G):

AGP(G)

AGP(G) generalizes **SSP**(G) (i.e. **SSP**(G) is **P**-time reducible to **AGP**(G)):



AGP(G) generalizes **BSMP**(G):



Question

Does $\mathbf{AGP}(G)$ reduce to $\mathbf{SSP}(G)$?

We don't know. But in all G with $\mathbf{P}$-time $\mathbf{SSP}(G)$ that we know, $\mathbf{AGP}(G)$ is also $\mathbf{P}$-time, by essentially the same arguments:

- $\mathbf{AGP}(\text{virtually f.g. nilpotent}) \in \mathbf{P}$ by polynomial growth,
- $\mathbf{AGP}(\text{hyperbolic}) \in \mathbf{P}$ by logarithmic depth of Van Kampen diagrams.

AGP(G)

Question

Does **AGP**(G) reduce to **SSP**(G)?

We don't know. But in all G with **P**-time **SSP**(G) that we know, **AGP**(G) is also **P**-time, by essentially the same arguments:

- **AGP**(virtually f.g. nilpotent) $\in$ **P** by polynomial growth,
- **AGP**(hyperbolic) $\in$ **P** by logarithmic depth of Van Kampen diagrams.

$\mathbf{AGP}(G)$

Question

Does $\mathbf{AGP}(G)$ reduce to $\mathbf{SSP}(G)$?

We don't know. But in all G with $\mathbf{P}$-time $\mathbf{SSP}(G)$ that we know, $\mathbf{AGP}(G)$ is also $\mathbf{P}$-time, by essentially the same arguments:

- $\mathbf{AGP}$(virtually f.g. nilpotent) $\in \mathbf{P}$ by polynomial growth,
- $\mathbf{AGP}$(hyperbolic) $\in \mathbf{P}$ by logarithmic depth of Van Kampen diagrams.

Question

Does **AGP**(G) reduce to **SSP**(G)?

We don't know. But in all G with **P**-time **SSP**(G) that we know, **AGP**(G) is also **P**-time, by essentially the same arguments:

- **AGP**(virtually f.g. nilpotent) $\in$ **P** by polynomial growth,
- **AGP**(hyperbolic) $\in$ **P** by logarithmic depth of Van Kampen diagrams.

$\mathbf{AGP}(G)$

Question

Does $\mathbf{AGP}(G)$ reduce to $\mathbf{SSP}(G)$?

We don't know. But in all G with $\mathbf{P}$-time $\mathbf{SSP}(G)$ that we know, $\mathbf{AGP}(G)$ is also $\mathbf{P}$-time, by essentially the same arguments:

- $\mathbf{AGP}$(virtually f.g. nilpotent) $\in \mathbf{P}$ by polynomial growth,
- $\mathbf{AGP}$(hyperbolic) $\in \mathbf{P}$ by logarithmic depth of Van Kampen diagrams.

AGP plays nicely with free products:

Theorem

Let G, H be finitely generated groups. Then **AGP**($G * H$) is **P**-time Cook reducible to **AGP**(G), **AGP**(H).

Proof: same as what we tried to do with **SSP**, only this time it works.

AGP plays nicely with free products:

Theorem

Let G, H be finitely generated groups. Then $\mathbf{AGP}(G * H)$ is **P**-time Cook reducible to $\mathbf{AGP}(G), \mathbf{AGP}(H)$.

Proof: same as what we tried to do with **SSP**, only this time it works.

Corollary

If G, H are finitely generated groups such that **AGP**(G), **AGP**$(H) \in$ **P** then **AGP**$(G * H) \in$ **P**.

Corollary

SSP, **BKP**, **BSMP**, **AGP** are polynomial time decidable in free products of finitely generated virtually nilpotent and hyperbolic groups in any finite number.

AGP($G * H$)

Corollary

If G, H are finitely generated groups such that **AGP**(G), **AGP**(H) $\in$ **P** then **AGP**($G * H$) $\in$ **P**.

Corollary

SSP, **BKP**, **BSMP**, **AGP** are polynomial time decidable in free products of finitely generated virtually nilpotent and hyperbolic groups in any finite number.

What about Knapsack Problem **KP**$(G * H)$?

Difficulty: put a bound on exponents n_i in

$$g_1^{n_1} \dots g_k^{n_k} = g.$$

We can do it in

- abelian groups (by linear algebra),
- hyperbolic groups (thin n-gons).

KP and free products

What about Knapsack Problem $\mathbf{KP}(G * H)$?

Difficulty: put a bound on exponents n_i in

$$g_1^{n_1} \dots g_k^{n_k} = g.$$

We can do it in

- abelian groups (by linear algebra),
- hyperbolic groups (thin n-gons).

What about Knapsack Problem $\mathbf{KP}(G * H)$?

Difficulty: put a bound on exponents n_i in

$$g_1^{n_1} \dots g_k^{n_k} = g.$$

We can do it in

- abelian groups (by linear algebra),
- hyperbolic groups (thin n-gons).

KP and free products

What about Knapsack Problem $\mathbf{KP}(G * H)$?

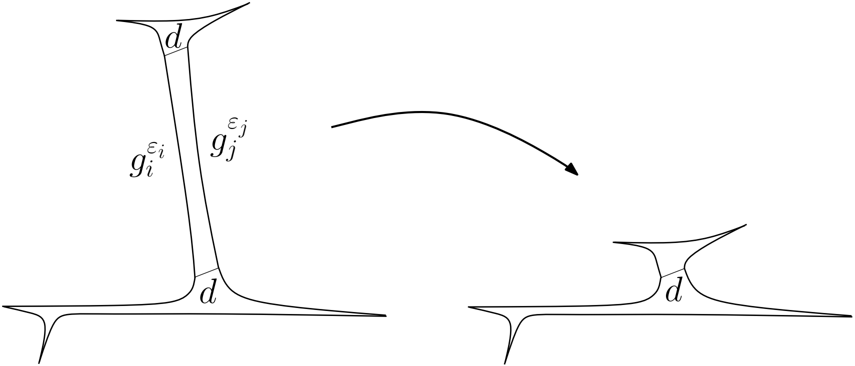
Difficulty: put a bound on exponents n_i in

$$g_1^{n_1} \dots g_k^{n_k} = g.$$

We can do it in

- abelian groups (by linear algebra),
- hyperbolic groups (thin n-gons).

In hyperbolic groups:



KP and free products

Similar argument works in free products, which gives

Theorem

If G, H are groups such that $\mathbf{KP}(G), \mathbf{KP}(H) \in \mathbf{P}$, then $\mathbf{KP}(G * H)$ is $\mathbf{P}$-time reducible to $\mathbf{BKP}(G * H)$.

Corollary

If G, H are groups such that $\mathbf{AGP}(G)$, $\mathbf{AGP}(H) \in \mathbf{P}$ and $\mathbf{KP}(G)$, $\mathbf{KP}(H) \in \mathbf{P}$ then $\mathbf{KP}(G * H) \in \mathbf{P}$.

Corollary

KP is polynomial time decidable in free products of finitely generated abelian and hyperbolic groups in any finite number.

Similar argument works in free products, which gives

Theorem

If G, H are groups such that $\mathbf{KP}(G), \mathbf{KP}(H) \in \mathbf{P}$, then $\mathbf{KP}(G * H)$ is $\mathbf{P}$-time reducible to $\mathbf{BKP}(G * H)$.

Corollary

If G, H are groups such that $\mathbf{AGP}(G)$, $\mathbf{AGP}(H) \in \mathbf{P}$ and $\mathbf{KP}(G)$, $\mathbf{KP}(H) \in \mathbf{P}$ then $\mathbf{KP}(G * H) \in \mathbf{P}$.

Corollary

KP is polynomial time decidable in free products of finitely generated abelian and hyperbolic groups in any finite number.

KP and free products

Similar argument works in free products, which gives

Theorem

If G, H are groups such that $\mathbf{KP}(G), \mathbf{KP}(H) \in \mathbf{P}$, then $\mathbf{KP}(G * H)$ is $\mathbf{P}$-time reducible to $\mathbf{BKP}(G * H)$.

Corollary

If G, H are groups such that $\mathbf{AGP}(G)$, $\mathbf{AGP}(H) \in \mathbf{P}$ and $\mathbf{KP}(G)$, $\mathbf{KP}(H) \in \mathbf{P}$ then $\mathbf{KP}(G * H) \in \mathbf{P}$.

Corollary

KP is polynomial time decidable in free products of finitely generated abelian and hyperbolic groups in any finite number.

SSP and direct products

AGP$(G \times H)$ is decidable whenever **WP**(G), **WP**(H) are decidable. What about complexity?

AGP$(F_2 \times F_2)$ is **NP**-complete since **BSMP**$(F_2 \times F_2)$ is, by a variation of Mikhailova construction.

By itself, this does not mean **SSP**$(F_2 \times F_2)$ is **NP**-complete because we don't know whether **AGP**(G) reduces to **SSP**(G).

Question

Is **SSP**$(F_2 \times F_2)$ **NP**-complete?

Answer: we don't know... but we know about **SSP**$(F_2 \times F_2 \times \mathbb{Z})$!

SSP and direct products

AGP$(G \times H)$ is decidable whenever **WP**(G), **WP**(H) are decidable. What about complexity?

AGP$(F_2 \times F_2)$ is **NP**-complete since **BSMP**$(F_2 \times F_2)$ is, by a variation of Mikhailova construction.

By itself, this does not mean **SSP**$(F_2 \times F_2)$ is **NP**-complete because we don't know whether **AGP**(G) reduces to **SSP**(G).

Question

Is **SSP**$(F_2 \times F_2)$ **NP**-complete?

Answer: we don't know... but we know about **SSP**$(F_2 \times F_2 \times \mathbb{Z})$!

SSP and direct products

AGP$(G \times H)$ is decidable whenever **WP**(G), **WP**(H) are decidable. What about complexity?

AGP$(F_2 \times F_2)$ is **NP**-complete since **BSMP**$(F_2 \times F_2)$ is, by a variation of Mikhailova construction.

By itself, this does not mean **SSP**$(F_2 \times F_2)$ is **NP**-complete because we don't know whether **AGP**(G) reduces to **SSP**(G).

Question

Is **SSP**$(F_2 \times F_2)$ **NP**-complete?

Answer: we don't know... but we know about **SSP**$(F_2 \times F_2 \times \mathbb{Z})$!

AGP$(G \times H)$ is decidable whenever **WP**(G), **WP**(H) are decidable. What about complexity?

AGP$(F_2 \times F_2)$ is **NP**-complete since **BSMP**$(F_2 \times F_2)$ is, by a variation of Mikhailova construction.

By itself, this does not mean **SSP**$(F_2 \times F_2)$ is **NP**-complete because we don't know whether **AGP**(G) reduces to **SSP**(G).

Question

Is **SSP**$(F_2 \times F_2)$ **NP**-complete?

Answer: we don't know... but we know about **SSP**$(F_2 \times F_2 \times \mathbb{Z})$!

SSP and direct products

AGP$(G \times H)$ is decidable whenever **WP**(G), **WP**(H) are decidable. What about complexity?

AGP$(F_2 \times F_2)$ is **NP**-complete since **BSMP**$(F_2 \times F_2)$ is, by a variation of Mikhailova construction.

By itself, this does not mean **SSP**$(F_2 \times F_2)$ is **NP**-complete because we don't know whether **AGP**(G) reduces to **SSP**(G).

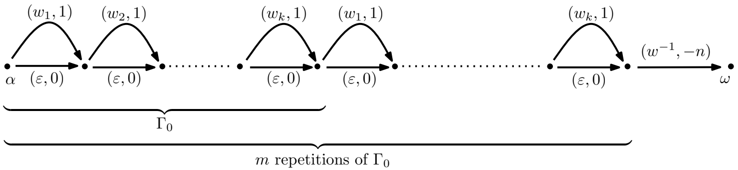
Question

Is **SSP**$(F_2 \times F_2)$ **NP**-complete?

Answer: we don't know... but we know about **SSP**$(F_2 \times F_2 \times \mathbb{Z})$!

BSMP(G) vs SSP($G \times \mathbb{Z}$)

BSMP(G) reduces to **SSP**($G \times \mathbb{Z}$):

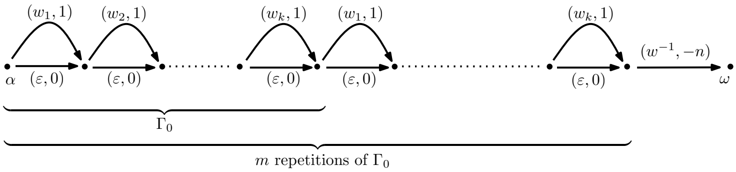


Corollary

SSP($F_2 \times F_2 \times \mathbb{Z}$) is **NP**-complete.

BSMP(G) vs SSP($G \times \mathbb{Z}$)

BSMP(G) reduces to **SSP**($G \times \mathbb{Z}$):



Corollary

SSP($F_2 \times F_2 \times \mathbb{Z}$) is **NP**-complete.

SSP and direct products

Observation: $\mathbf{AGP}(G)$ and $\mathbf{AGP}(G \times \mathbb{Z})$ are **P**-time equivalent.

Corollary

There are groups G, H such that $\mathbf{SSP}(G), \mathbf{SSP}(H) \in \mathbf{P}$, but $\mathbf{SSP}(G \times H)$ is **NP**-complete.

Proof: $G = F_2$, $H = F_2 \times \mathbb{Z}$.

Observation: $\mathbf{AGP}(G)$ and $\mathbf{AGP}(G \times \mathbb{Z})$ are $\mathbf{P}$-time equivalent.

Corollary

There are groups G, H such that $\mathbf{SSP}(G), \mathbf{SSP}(H) \in \mathbf{P}$, but $\mathbf{SSP}(G \times H)$ is $\mathbf{NP}$-complete.

Proof: $G = F_2$, $H = F_2 \times \mathbb{Z}$.

SSP and direct products

Observation: $\mathbf{AGP}(G)$ and $\mathbf{AGP}(G \times \mathbb{Z})$ are $\mathbf{P}$-time equivalent.

Corollary

There are groups G, H such that $\mathbf{SSP}(G), \mathbf{SSP}(H) \in \mathbf{P}$, but $\mathbf{SSP}(G \times H)$ is $\mathbf{NP}$-complete.

Proof: $G = F_2$, $H = F_2 \times \mathbb{Z}$.

Open questions

Some of (many) open questions:

- Are there groups with decidable **WP** but undecidable **KP**?
- Is **KP**(f.g. nilpotent) decidable? If yes, is it in **P**?
- Is **SSP**(lamplighter) in **P**?
- Is **SSP**(polycyclic) in **P**?
- Is decidability of **KP** a geometric property? (Finite extensions and f.i. subgroups are fine.)
- What about **SSP**($G *_A H$), **SSP**(HNN)? (Finite amalgamated subgroups are fine.)
- What about **SSP**(relatively hyperbolic)?

Some of (many) open questions:

- Are there groups with decidable **WP** but undecidable **KP**?
- Is **KP**(f.g. nilpotent) decidable? If yes, is it in **P**?
- Is **SSP**(lamplighter) in **P**?
- Is **SSP**(polycyclic) in **P**?
- Is decidability of **KP** a geometric property? (Finite extensions and f.i. subgroups are fine.)
- What about **SSP**($G *_A H$), **SSP**(HNN)? (Finite amalgamated subgroups are fine.)
- What about **SSP**(relatively hyperbolic)?

Some of (many) open questions:

- Are there groups with decidable **WP** but undecidable **KP**?
- Is **KP**(f.g. nilpotent) decidable? If yes, is it in **P**?
- Is **SSP**(lamplighter) in **P**?
- Is **SSP**(polycyclic) in **P**?
- Is decidability of **KP** a geometric property? (Finite extensions and f.i. subgroups are fine.)
- What about **SSP**($G *_A H$), **SSP**(HNN)? (Finite amalgamated subgroups are fine.)
- What about **SSP**(relatively hyperbolic)?

Open questions

Some of (many) open questions:

- Are there groups with decidable **WP** but undecidable **KP**?
- Is **KP**(f.g. nilpotent) decidable? If yes, is it in **P**?
- Is **SSP**(lamplighter) in **P**?
- Is **SSP**(polycyclic) in **P**?
- Is decidability of **KP** a geometric property? (Finite extensions and f.i. subgroups are fine.)
- What about **SSP**($G *_A H$), **SSP**(HNN)? (Finite amalgamated subgroups are fine.)
- What about **SSP**(relatively hyperbolic)?

Open questions

Some of (many) open questions:

- Are there groups with decidable **WP** but undecidable **KP**?
- Is **KP**(f.g. nilpotent) decidable? If yes, is it in **P**?
- Is **SSP**(lamplighter) in **P**?
- Is **SSP**(polycyclic) in **P**?
- Is decidability of **KP** a geometric property? (Finite extensions and f.i. subgroups are fine.)
- What about **SSP**($G *_A H$), **SSP**(HNN)? (Finite amalgamated subgroups are fine.)
- What about **SSP**(relatively hyperbolic)?

Open questions

Some of (many) open questions:

- Are there groups with decidable **WP** but undecidable **KP**?
- Is **KP**(f.g. nilpotent) decidable? If yes, is it in **P**?
- Is **SSP**(lamplighter) in **P**?
- Is **SSP**(polycyclic) in **P**?
- Is decidability of **KP** a geometric property? (Finite extensions and f.i. subgroups are fine.)
- What about **SSP**($G *_A H$), **SSP**(HNN)? (Finite amalgamated subgroups are fine.)
- What about **SSP**(relatively hyperbolic)?

Open questions

Some of (many) open questions:

- Are there groups with decidable **WP** but undecidable **KP**?
- Is **KP**(f.g. nilpotent) decidable? If yes, is it in **P**?
- Is **SSP**(lamplighter) in **P**?
- Is **SSP**(polycyclic) in **P**?
- Is decidability of **KP** a geometric property? (Finite extensions and f.i. subgroups are fine.)
- What about **SSP**($G *_A H$), **SSP**(HNN)? (Finite amalgamated subgroups are fine.)
- What about **SSP**(relatively hyperbolic)?

Some of (many) open questions:

- Are there groups with decidable **WP** but undecidable **KP**?
- Is **KP**(f.g. nilpotent) decidable? If yes, is it in **P**?
- Is **SSP**(lamplighter) in **P**?
- Is **SSP**(polycyclic) in **P**?
- Is decidability of **KP** a geometric property? (Finite extensions and f.i. subgroups are fine.)
- What about **SSP**($G *_A H$), **SSP**(HNN)? (Finite amalgamated subgroups are fine.)
- What about **SSP**(relatively hyperbolic)?