1. (6pts)

Use a truth table to check if the following is a tautology.

\[(p \land q) \lor \neg p \iff \neg p \land q\]
2 (8pts) Are the following true or false? Explain.

- $\forall x \in R \exists y \in R \ [x \cdot y = 1]$

- $\forall x \in [0, \infty) \exists y \in R \ [y^2 = x]$

- $\exists x \in Z \forall y \in Z \ [\frac{y}{x} \in Z]$

- $\forall u \in (N - \{0\}) \exists v \in (N - \{u\}) \ [\frac{v}{u} \in N]$
3 (8pts) Let \( a \) be any **nonzero** rational number. Let \( b \) be an irrational number. Prove that \( a \cdot b \) is irrational.
1. Define: \( A \) is a **countably infinite set**.

2. Prove that the integers, \( \mathbb{Z} \), form a countably infinite set.

3. Let \( A = \{2^x \mid x \in \mathbb{Z}\} \). Prove or disprove: \( A \) is a countably infinite set.
5 (12pts) Let $A, B, C$ be subsets of a universe $U$. Prove using the element method or disprove with a counterexample.

1. If $A \cap (B - (A \cup C)) = \emptyset$

2. $(B \cup C) \times (A \cap B) \subseteq C \times A$

3. $A - (B - C) \subseteq (A - B) - C$
Let $A$ be a set of 30 balls, numbered 1 – 30. **Use the Pigeonhole Principle** to show that if 18 balls are drawn, some pair will sum to 35.
1. Are the following functions injections (1-1)? surjections (onto)? If not, explain.

1. \( f : \mathbb{Q} \rightarrow \mathbb{Q} \)  
   \( x \rightarrow x^2 \)

2. \( f : \mathbb{N} - \{0\} \rightarrow \mathbb{N} \)  
   \( n \rightarrow \lfloor \frac{n^2 + 3}{n} \rfloor \)

3. \( f : \{n \in \mathbb{N} \mid n \text{ is even}\} \rightarrow \mathbb{N} \)  
   \( n \rightarrow \text{the sum of the digits of } n \)

2. Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \) be injective (1-1) functions. Prove that \( g \circ f : A \rightarrow C \) is an injective (1-1) function.

   \textit{Hint:} Get the first step right and the last step right!
Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f^{-1} : \mathbb{R} \to \mathbb{R}$ is also a function.

Prove or disprove:

1. $f$ must be 1–1.

2. $f$ must be onto.
9 (12pts)

1. Let \( A \) be a finite set with \( n \geq 1 \) elements.

   • How many relations are there on \( A \)?

   • How many functions \( f : A \to A \) are there?

   • How many onto functions \( f : A \to A \) are there?

2. Let \( A = \{1, 2, 3\} \). Find all functions \( f : A \to A \) such that \( f^2 = f_i \), where \( f_i \) is the identity function.
Let $f : R \rightarrow R$ be given by $f(x) = x^3 - 1$ and $g : R \rightarrow R$ be given by $g(x) = x + 2$. Let $A = [-1, 1]$ and $B = [0, \infty)$. Find

1. $f(A)$

2. $f^{-1}(B)$

3. Solve $(gof)(x) - 1 = 0$. 
11 (10pts)

1. Let $R \subseteq A \times A$ be a symmetric and antisymmetric relation on a set $A$. Prove that $R$ is transitive.

2. Let $R \subseteq A \times A$ be an antisymmetric relation on a set $A$. Prove that $R^{-1}$ is an antisymmetric relation on $A$.

3. Let $A = \{1, 2, 3, 4\}$ and $R$ be a relation on $A$ that is reflexive, symmetric and antisymmetric. Find $R$. 
12 (12pts) Decide if each relation on the given set $A$ is reflexive, symmetric, antisymmetric and/or transitive. If not, explain.

1. $A = R$. $R_1 : \{(x, y) \mid x^2 + y^2 \leq 2\}$.

2. $A = Z$. $R_2 : \{(x, y) \mid x \equiv y \pmod{5}\}$.

3. $A = \Sigma^*$, where $\Sigma = \{0, 1\}$. $R_3 : \{(w_1, w_2) \mid \text{the number of 0's in } w_1 = \text{the number of 1's in } w_2\}$. 
• Define an equivalence relation $R$ on $A = \{1, 2, 3\}$ in which $(1, 3) \notin R$.

• Let $R$ be the relation defined on points in the plane given by $(a, b) R (c, d)$ if and only if $(a - 2)^2 + (b - 1)^2 = (c - 2)^2 + (d - 1)^2$.

1. Prove that $R$ is an equivalence relation.

2. Describe $[(6, 4)]$ geometrically.
Consider the relations $R$ and $S$ given below.

Find

- the symmetric closure $S^s$.
- the transitive closure $S^t$.
- $RS$
- $(R \cap S)^t$
1. Let \((A, R)\) be a poset on \(A = \{1, 2, 3, 4, 5, 6\}\). The Hasse diagram \(H\) for \(R\) is given below.

- Find all maximal elements.

- Is \(R\) a total ordering? Explain.

- Draw the entire relation \(R\).

2. Draw the Hasse diagram for a poset \((B, S)\) which contains both

   (a) a unique maximal element \(b\), and

   (b) an element \(z \in B\) that is not comparable to \(b\).

   *Hint:* \(B\) need not be finite.
Let $S$ and $T$ be finite nonempty sets. Prove by induction: $\frac{|S| + |T|}{2} \leq |S \cup T|$.
17 (10pts) Let \( n \in \mathbb{N} \) be a natural number having 2 or more digits. Prove by induction: the product of the digits is not more than \( n \).
18 (10pts) Prove by induction: \( \forall n \geq 1, \) the complete bipartite graph on \( 2n \) vertices, \( K_{n,n} \), has \( n^2 \) edges.
Find, if possible, a connected graph on 9 vertices with the following properties.

1. vertex degrees $(4, 3, 3, 3, 3, 2, 2, 2, 2)$ and not hamiltonian.

2. vertex degrees $(4, 4, 4, 4, 2, 2, 2, 2)$ and eulerian.

3. a tree with exactly 2 vertices of degree 3 that are adjacent.

4. planar, hamiltonian, exactly one vertex of degree 6 and no vertices of degree 2.
1. Draw a connected plane graph $G$ on 9 vertices that has exactly 21 edges. Clearly number the edges.

2. Let $G$ be a connected graph on $n$ vertices having exactly one cycle. How many edges does $G$ have? Explain. 

*Hint*: You can quote any theorem we proved in class.

3. Prove or disprove: $f(n) = 2^n$ is $O(n!)$.

4. Prove or disprove: $f(n) = n \ln n$ is $O(n)$. 