1. (6pts) Is the following argument valid? **Use a truth table** to prove or disprove.

- P1: If I work hard, then I earn lots of money.
- P2: If I earn lots of money, then I pay high taxes.
- C: If I do not work hard, then I do not pay high taxes.
2 (16pts) True or false? If false, explain.
Let $U = Z$ in 1,2,and 3.

1. $\forall x \exists ! y [xy = 0]$
2. $\forall x \exists y [x + 2y = 3]$
3. $\exists y \forall x [x + y = 30]$
4. $Q \cap R^+ = Q$
5. $Q \cup Z = Q$
6. $\{3\} \subseteq \{1, 2, 3\}$
7. $1 \in \{\{1\}, 2\}$
8. $\{1\} \cap \{2\} = \emptyset$
1. Define: A set $X$ is a countably infinite set.

2. Outline an argument to prove that the positive rational numbers, $\mathbb{Q}^+$, form a countably infinite set.
4 (8pts)
Prove that if the average of 4 different integers is 10, then at least one of the integers is greater than 11.
Let $A, B, C$ be subsets of a universe $U$. Prove using the element method or disprove with a counterexample.

1. If $A \subseteq B$, then $A - B = B - A$.

2. $(A \times B) \cap (B \times C) \subseteq (A \times C)$.

3. $(A - B) \cup (B - C) \subseteq A - C$. 
(8pts) **Use the Pigeonhole Principle** to prove that given any 7 positive integers, there are 2 whose sum or difference is divisible by 10.

*Note*: Show how you are using the Pigeonhole Principle.
Let

\[ f : \mathbb{R}^+ \to (1, \infty) \]
\[ x \to \frac{x + 1}{x} \]

\[ g : \mathbb{Q} \to \mathbb{Q} \]
\[ x \to x^2 + 5x + 10 \]

\[ h : [1, \infty) \to [1, \infty) \]
\[ x \to \frac{1}{x + 1} \]

\[ i : P(S) \to P(S), \text{ where } S = \{a, b, c\} \]
\[ A \to S - A \]

1. Is \( f \) 1-1? onto? Explain.

2. Is \( g \) 1-1? onto? Explain.


8 (12pts) Let \( A = \{1, 2, 3, 4\} \), and

\[
f : A \rightarrow A \times A
\]
\[
a \rightarrow (a, a)
\]
\[
g : A \times A \rightarrow A
\]
\[
(x, y) \rightarrow x.
\]

1. How many relations are there on \( A \)?

2. How many reflexive relations are there on \( A \)?

3. How many functions have domain \( A \) and codomain \( A \times A \)?

4. How many \textbf{1-1} functions have domain \( A \) and codomain \( A \times A \)?

5. Find \( g \circ f \)

6. Find \( f \circ g \)
Let the functions $f, g : \mathbb{N} \to \mathbb{N}$ be given by $f(n) = n + 2$ and $g(n) = n^4$. Find

- $(g \circ f)(3)$
- $f^2(2)$
- $g(A)$, where $A = \{-1, 0, 1, 2\}$
- $f^{-1}(B)$, where $B = \{10, 20\}$.

Let $A = \{1, 2, 3, 4\}$.

- Define an equivalence relation $R$ on $A$ in which $(1, 4) \in R$ and $(2, 3) \notin R$.

- What is $[2]$ in your relation?
10 (12pts)

1. Let $R_1$ and $R_2$ be antisymmetric relations on a set $A$. Either prove or disprove with a counterexample.

   (a) $R_1 \cup R_2$ is antisymmetric.

   (b) $R_1 \cap R_2$ is antisymmetric.

   (c) $R_1^2$ is antisymmetric.

2. Let $R_1$ and $R_2$ be reflexive relations on a set $A$. Either prove that $R_1 \cup R_2$ is reflexive or disprove with a counterexample.
11 (16pts) Decide if each of the following relations on the given set $A$ is reflexive, symmetric, antisymmetric and/or transitive. If not, explain.

1. $A = \mathbb{Z}$
   $$R_1 : \{(x, y) \mid x + 2y \text{ is even}\}$$

2. $A = \{0, 1, 2, 3, 4, 5, 6\}$
   $$R_2 : \{(x, y) \mid (x - 2.5) \cdot (y - 2.5) \geq 0\}$$

3. $A = \mathbb{Z}$
   $$R_3 : \{(x, y) \mid |x - y| \leq 2\}$$

4. $S = \{1, 2, 3\}$ and $A, B \in P(S)$.
   $$R_4 : \{(A, B) \mid A \cap B \neq \emptyset\}$$
Let $A = \mathbb{R}$ and $S$ be a relation on $A$ given by $S: \{ (x, y) \mid \sin^2 x + \cos^2 y = 1 \}$.

1. **Prove** that $S$ is an equivalence relation.

2. What is the equivalence class of $\pi/2$?

3. Is $S$ antisymmetric? **Explain.**
13 (12pts) Let $A = \{0, 1, 2, 3, 4\}$ and $R$ be the relation on $A$ given by 
$R: \{(0, 1), (1, 2), (0, 2), (3, 4), (4, 3)\}$.

Find

1. $R^{-1}$

2. $R^s$

3. $R^t$

4. $R^2$
Let $A = \{2, 3, 4, 6, 8, 12, 24, 30, 33, 48, 60, 90\}$ and $R$ be the “divides” relation on $A$.

1. Draw the Hasse diagram for $R$.

2. Find all maximal elements, if any.

3. Find all minimal elements, if any.

4. Is $R$ a total ordering on $A$? Explain.

5. Partition the elements of $A$ into the smallest possible number of antichains.
15 (8pts) Let $x \in \mathbb{R}$, $x \neq 1$.

Prove by induction: $\forall \ n \geq 0, \ \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$. 
1. Let $R \subseteq A \times A$. Prove by induction that for all positive integers $s, t$,

$$R^{s+t} = R^s R^t.$$ 

*Hint:* Use the fact that the composition of relations is associative.

2. Prove by induction: $\forall \ n \geq 1, \ 8 \mid 3^{2n} - 1.$
1. Suppose $G$ is a connected, simple graph on $n$ vertices having only 2 vertices, $u$ and $v$, of odd degree. If $u$ and $v$ are not adjacent, find the length (number of edges) of a longest trail in $G$.

2. Is $K_{3,4}$ planar? Explain.

3. Is $f(n) = \frac{n^2 - 3n}{3} O(n^2)$? Prove or disprove.

4. Is $f(n) = \frac{n^2 - 2n}{5} O(n)$? Prove or disprove.
1. Find, if possible, a simple connected graph on 10 vertices and 23 edges that is not hamiltonian and not eulerian.

2. Find, if possible, a simple connected graph on 10 vertices and 23 edges that is hamiltonian and eulerian.

3. Find, if possible, a tree having exactly 2 vertices of degree 1 no vertices of degree 3.

4. Find, if possible, a planar connected graph having 5 vertices and 6 edges.