1. (6pts)

Use a truth table to check if the following is a tautology.

\[ ((p \rightarrow q) \rightarrow r) \leftrightarrow (p \rightarrow (q \rightarrow r)) \]
2 (10pts) Find the negation without using the negation symbol.

- $\exists x \left[ \forall y(xy = y) \right]$ 

- $\forall x (x \text{ is prime} \rightarrow x^2 + 1 \text{ is even})$.

Are the following true or false? Explain.

- $U = \mathbb{R}, \; \forall x \; \exists y \left[ x \cdot y = 1 \right]$ 

- $U = \mathbb{R}^+, \; \forall x \; \exists y \left[ y^2 = x \right]$ 

- $U = \mathbb{Z}, \; \exists x \; \forall y \left[ \frac{y}{x} \in \mathbb{Z} \right]$
3 (8pts) Let $a$ be any rational number and $b$ be any irrational number. Prove that $a + b$ is irrational.
4 (10pts)

1. Define: $A$ is a \textbf{countably infinite set}.

2. Prove that the integers form a countably infinite set.

3. Let $A$ and $B$ be a countably infinite sets. Suppose $A \cap B = \emptyset$.

   Prove: $A \cup B$ is a countably infinite set.
5 (12pts) Let $A, B, C$ be subsets of a universe $U$. Prove using the **element method** or disprove with a **counterexample**.

1. $A \times (B - C) \subseteq (A \times B) - (A \times C)$
2. $(A \cap B) \cap (A - B) = \emptyset$
3. $A - (B - C) = (A \cap B) - C$
6 (8pts) **Use the Pigeonhole Principle** to find the smallest number of integers between 200 and 999 that must be chosen so that at least 2 of them have a digit in common.
1. Are the following functions injections (1-1)? surjections (onto)? If not, explain.

1. \( f : \mathbb{Q} - \{1\} \rightarrow \mathbb{Q} \)
   \[ r \rightarrow \frac{r}{1-r} \]

2. \( f : \mathbb{Z} \rightarrow \mathbb{N} \)
   \[ n \rightarrow |n| + 1 \]

3. \( f : \mathbb{R} \rightarrow \mathbb{R} \)
   \[ x \rightarrow 4 - 2x \]
8 (8pts)
Let $X = \{1, 2, 3, 4\}$.

1. Find a function $g : X \to X$ with $g(3) = 4$ and such that $g^2 = g^{-1}$.

2. Find an equivalence relation on $X$ with 2 equivalence classes.
9 (12pts)

Let \( X = \{1, 2\}, Y = \{1, 2, 3\}, Z = \{1, 2, 3, 4\} \)

1. How many relations are there on \( X \)?

2. How many functions are there from \( Z \) to \( Z \)?

3. How many onto functions are there from \( Y \) to \( Z - \{3\} \)?

4. List all 1 – 1 functions from \( X \) to \( Y \).
1. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be given by $f(m) = m + 1$ and $h : \mathbb{Z} \rightarrow \mathbb{Z}$ be given by

$$h(m) = \begin{cases} 
0 & \text{if } m \text{ is even} \\
1 & \text{if } m \text{ is odd}.
\end{cases}$$

Find

- $h \circ f$
- $f \circ h$
- All solutions to $f \circ h = h \circ f$

2. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be onto functions. Prove that $g \circ f : A \rightarrow C$ is an onto function.

*Hint*: Get the first step right and the last step right!
11 (12pts) Let $R_1$ and $R_2$ be transitive relations on a set $A$ and $S_1$ and $S_2$ be antisymmetric relations on $A$. Either prove or disprove with a counterexample.

1. $R_1 \cup R_2$ is transitive

2. $R_1 - R_2$ is transitive

3. $S_1 \cup S_2$ is antisymmetric

4. $S_1 - S_2$ is antisymmetric
12 (12pts) Decide if each relation on the given set $A$ is reflexive, symmetric, antisymmetric and/or transitive. If not, explain.

1. $A = R$. $R_1 : \{(x, y) \mid x^3 + y^2 \geq 4\}$.

2. $A = \mathbb{Z}$. $R_2 : \{(m, n) \mid m = |n|\}$.

3. $A = \Sigma^*$, where $\Sigma = \{0, 1\}$. $R_3 : \{(w_1, w_2) \mid \text{the number of 0's in } w_1 = \text{twice the number of 1's in } w_2\}$. 
13 (12pts)

1. Let $f : X \to Y$ and $R$ be a relation on $X$ given by $R : \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$.

   - Prove that $R$ is an equivalence relation on $X$.
   - Describe the equivalence classes of $R$.

2. Let $A = \{1, 2, 3, 4, 5, 6\}$. Define two equivalence relations on $A$, $R_1$ and $R_2$, for which the following 3 properties simultaneously hold:

   - $(1, 2) \in R_1$, and $(4, 5) \not\in R_1$
   - $(1, 2) \not\in R_2$
   - $R_1 \cup R_2$ is not an equivalence relation.
Let $A = \{-3, -2, -1, 0, 1, 2, 3\}$ and consider the relation $R$ on $A$ given by $R = \{(m, n) \mid n = m + 1\}$.

Find

- the reflexive closure $R^r$.

- the symmetric closure $R^s$.

- the transitive closure $R^t$.

- $R^2$

- $R^{-1}$
15 (8pts) Let $A = \{1, 2, 3, 4\}$ and $B \subseteq P(A)$, where
$B = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$. Let $R$ be the poset on $B$ given by subset inclusion.

1. Draw the Hasse diagram for $(B, R)$.

2. Find all maximal elements.

3. Partition $B$ into the smallest possible number of antichains.
16 (8pts) A chessboard is called **defective** if one square is missing.

Prove **by induction**: For all $n \geq 1$, a $2^n$ by $2^n$ defective chessboard can be tiled using the “L” shaped figures below.
17 (8pts) Prove by induction: \( \forall n \geq 1, 9 \mid n^3 + (n + 1)^3 + (n + 2)^3 \).
18 (8pts) Prove by induction: \( \forall n \geq 1, \)

\[
\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}.
\]
19 (12pts)
Find, if possible, a connected graph with the following properties.

1. a tree that is not bipartite.

2. nonhamiltonian with 10 vertices and 25 edges.

3. eulerian with an odd number of edges.

4. planar, nonhamiltonian, and bipartite with 5 vertices.
1. Define: Graphs $G_1$ and $G_2$ are isomorphic.

2. Find all nonisomorphic trees having 5 vertices.

3. Prove or disprove: $f(n) = 1 + 2 + 3 + \cdots n$ is $O(n^2)$.

4. Prove or disprove: $g(n) = (n^2 + 8)(n + 1)$ is $O(n^3)$.