1. (6pts)

Use a truth table to determine if the following is a tautology.

\[ p \rightarrow (q \lor r) \leftrightarrow [(p \rightarrow q) \lor (p \rightarrow r)] \]
2 (10pts) Let $U = \mathbb{Z}$. Are the following true or false? If false, explain.

1. \begin{itemize}
   \item $\forall x \exists! y [x - y = 0]$
   \item $\forall x \exists y \left[ \frac{x}{y} = 1 \right]$
\end{itemize}

2. **Prove** that $\sqrt{2}$ is irrational.

   *Hint:* Assume $\sqrt{2}$ is rational and reach a contradiction.
1. Define what it means for a set to be **countably infinite**.

2. Let $A$ be the set of all positive integers $n$ with the property that all of the digits of $n$ are different. Prove that $A$ is a countably infinite set.
4 (9pts) Either prove or disprove with a counterexample.

1. \(\forall n \geq 1, 7n + 2\) is a perfect square.

2. The square of any odd integer \(m \geq 1\) has the form \(8m + 1\) for some integer \(m\).

3. \(2^{66} - 1\) is a prime number.
5 (12pts) Let $A, B, C, D$ be subsets of a universe $U$. Prove using the element method or disprove with a counterexample.

1. $(A \cap B) \cup C = (A \cup C) \cap B$

2. $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

3. $(A \cup B) \cap (C \cup D) = (A \cap C) \cup (B \cap D)$
6 (10pts)

**Use the Pigeonhole Principle** to show that if $S$ is any set of 12 natural numbers, some two distinct numbers in $S$ will differ by a multiple of 11.

Show how you are using the Pigeonhole Principle.
1. Let \( f : A \to B \) and \( g : B \to C \) be onto functions. Prove that \( gof : A \to C \) is an onto function.

\[ \text{Hint: Get the first step right and the last step right!} \]

2. Let \( X = \{1, 2, 3, 4, 5\} \). Find a function \( f : X \to X \) such that \( f(2) = 2, f(4) = 4, f(5) = 3 \) and

(a) \( f^2(x) = f(x) \)

(b) \( f(x) \) is \textbf{not} onto.
• Find a set \( A \) and an onto function \( f : A \to A \) that is **not** a 1–1 function.

• Are the following functions 1-1? onto? If not, explain.

1. \( f : \mathbb{R} \to \mathbb{R} \)
   \[ x \to x + |x| \]

2. \( f : \mathbb{Z} \to \mathbb{Z} \)
   \[ n \to n^2 + 3 \]
1. Suppose $f(x)$ and $g(x)$ are real-valued functions defined on the same set of real numbers. Define 

**f is of order g, i.e., $f(x)$ is $O(g(x))$.**

2. Find a function $g(n)$ of the form $A^B$ such that $f(n)$ is $O(g(n))$ if

- $f(n) = \frac{5n^3 + 6}{n + 2}$
- $f(n) = n!$

**Prove your results**
1. Consider the function

\[ g : \mathbb{R} \rightarrow \mathbb{R} \]
\[ x \rightarrow x^3 \]

- Find \( g^{-1}\{\{-8, 27\}\} \).
- Find \( g^2(2) \)

2. Let \( X \) be a set. Find a \( 1-1 \) function \( f : X \rightarrow X \) that is not onto. Is \( f^{-1} : X \rightarrow X \) a function? Explain.
11 (10pts)

1. Let $R \subseteq A \times A$ be a transitive relation on a set $A$. Prove that $R^{-1}$ is a transitive relation on $A$.

   \textit{Hint:} Get the first step right and the last step right!

2. Let $S$ be an equivalence relation on a set $A$. \textbf{Prove or disprove:} $S^{-1}$ is an equivalence relation on $A$. 

12 (12pts) Decide if each of the following relations on the given set \( A \) is reflexive, symmetric, antisymmetric and/or transitive. If not, explain.

1. \( A = \{0, 1, 2\} \). \( R_1 = \{(0, 1), (1, 0), (2, 0), (2, 1)\} \).

2. \( A = R \). \( R_2 : \{(x, y) \mid x^2y^2 \geq 1\} \).

3. \( A = \Sigma^* \), where \( \Sigma = \{a, b\} \). \( R_3 : \{(w_1, w_2) \mid w_1^R = w_2\} \).

Note: \( w^R \) is the reverse of \( w \), e.g., \( aaba^R = abaa \).
13 (12pts) Let \( A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). Define a relation \( R \) on \( A \) as follows:

\[
m R n \text{ if and only if } m \cdot n \text{ is a perfect square.}
\]

- Explain why \( R \) is an equivalence relation. In particular, prove that \( R \) is transitive.

- Find all equivalence classes of \( R \).
Let $A = \{0, 1, 2, 3\}$. Consider the relation $R = \{(0, 0), (0, 1), (1, 0), (1, 2), (2, 0), (3, 0), (3, 2)\}$ on $A$. Sketch the relation $R$ and then find

- the symmetric closure $R^s$.
- the transitive closure $R^t$.
- $R^{-1}$
- $R^2$
15 (10pts) Let $A$ be the set of all positive integers $n$, $1 < n < 48$, that divide 48 evenly and $R$ be the relation on $A$ with $x \ R \ y$ if and only if $x$ divides evenly into $y$.

- Draw the Hasse diagram $H$ for the poset $(A, R)$.

- Find all maximal elements.

- Find all minimal elements.

- Partition $A$ into the smallest possible number of disjoint antichains.
16 (8pts) Let $b, c \in \mathbb{Z}$. Suppose $\sum_{r=1}^{n}(br + c) = f(n, b, c)$. Find $f(n, b, c)$ and prove your result by induction.

Prove your answer is correct by induction.
17 (8pts) Prove by induction: \( \forall n \geq 1, \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}. \)
18 (8pts) Prove by induction: $\forall n \geq 1, 6 \mid n^3 + 5n$. 
19 (18pts) Find, if possible, a simple graph with the following properties. If not possible, explain why not.

1. A planar bipartite graph in which each vertex has degree 4.

2. A graph with 8 vertices having degrees (4, 3, 2, 1, 1, 1, 1, 1). One vertex of degree 4.

3. A graph with 15 vertices, each with degree 5.

4. A connected non-eulerian, non-hamiltonian graph on 8 vertices in which no vertex has degree 3.

5. A graph with 11 vertices which is eulerian but not hamiltonian.

6. A graph with 20 vertices and 25 edges that is both hamiltonian and eulerian.
1. Find, if possible, a positive integer $t \geq 1$ such that the graph $K_{2,t}$ is not planar. Explain your answer.

2. Let $T$ be a tree with $n \geq 2$ vertices. Prove that $T$ has at least 2 vertices of degree 1.