1. (8pts)
Is the following a tautology? Prove or disprove.

\[ ((p \land q) \leftrightarrow q) \leftrightarrow (q \rightarrow p) \]

2 (12pts) Let \( U = \mathbb{Z} \). Are the following true or false? Explain.

1. \( \exists ! y \ \forall x \ [xy = 0] \)

2. \( \exists x \ \exists y \ [x > 1 \land xy = 1] \)

3. \( \exists x \ \forall y \ [\frac{y}{x} \in \mathbb{Z}] \)

Write the negation of 2 and 3 without using the negation symbol (\( \neg \)).
1. Define what it means for a set to be **countably infinite**.

2. Let \( X = \{1, 2\} \), \( Y = \mathbb{Z}^+ \), and let \( A \) be the set of all functions with domain \( X \) and codomain \( Y \). Show that \( A \) is a countably infinite set.

   *Hint:* Show how to list all the functions.
4 (8pts) Let $n \in \mathbb{Z}$. Prove that if $4|n^2$, then $n$ is even.
5 (12pts) Let $A, B, C$ be subsets of a universe $U$. Prove using the element method or disprove with a counterexample.

1. $A \subseteq (A \cap B) \cup (A - B)$

2. $(A \cap B) \cap (A - B) \neq \emptyset$

3. If $B \subseteq C$, then $(A \times B) \subseteq (A \times C)$
6 (10pts) Let \( A \) be a set of 5 points in the plane having integer coordinates. **Use the Pigeonhole Principle** to show that the midpoint of the line segment joining some pair of them has integer coordinates.

Show how you are using the Pigeonhole Principle.
7 (10pts) Let $f : A \to B$ and $g : B \to C$ be functions. Prove the following.

1. If $f$ and $g$ are onto functions, then $gof$ is an onto function.

2. If $gof$ is the identity function on $A$, then $f$ is $1-1$. 
8 (12pts) Let $A = [4]$. Find all bijections on $A$ having no fixed point, i.e., no $a \in A$ such that $f(a) = a$. 
1. Are the following functions injections (1-1)? surjections (onto)? If not, explain.

1. \( f : Q - \{2\} \rightarrow Q \)
\[ r \rightarrow \frac{r}{2-r} \]

2. \( f : Z \rightarrow N \)
\[ n \rightarrow |n| + 3 \]

3. \( f : R \rightarrow R \)
\[ x \rightarrow 4 - x \]
10 (12pts) Let $R_1, R_2 \subseteq A \times A$ be relations on a set $A$. Prove or disprove.

1. If $R_1$ and $R_2$ are symmetric, then $R_1 - R_2$ is symmetric.

2. If $R_1$ and $R_2$ are antisymmetric, then $R_1 - R_2$ is antisymmetric.
11 (12pts) Decide if each of the following relations on the given set $A$ is reflexive, symmetric, antisymmetric and/or transitive. If not, explain.

1. $A = [5]$. $R_1: \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$

2. $A = \mathbb{Z}$. $R_2: \{(m, n) \mid m \cdot n > 0\}$.

3. $A = \Sigma^*$, where $\Sigma = \{0, 1\}$. $R_3: \{(w_1, w_2) \mid w_1 \cdot w_2 \text{ has an even number of 0's.}\}$.
1. Let $S = \{a, b, c, d, e, f\}$ and consider the partition $\{\{a, b, c\}, \{d, e\}, \{f\}\}$ of $S$. What are the ordered pairs in the corresponding equivalence relation.

*Hint:* You can use pictures.

2. Consider the relation $S$ on the real numbers $R$ as follows:

$$x \ S \ y \text{ if and only if } |x| = |y|.$$ 

Explain why $S$ is an equivalence relation. What are the equivalence classes of $S$?
Let \( A = \{0, 1, 2, 3\} \). Consider the relation \( S = \{(0, 1), (1, 0), (1, 2), (2, 3)\} \) on \( A \). Sketch the relation \( S \) and then find

- the symmetric closure \( S^s \).

- the transitive closure \( S^t \).

- \( S^{-1} \)

- \( S^2 \)
14 (10pts) Let $A = \{1, 2, 3, 4\}$ and $B \subseteq P(A)$, where
$B = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{1, 2, 4\}, \{1, 2, 3\}\}$. Let $R$ be the poset on $B$ given by subset inclusion.

1. Draw the Hasse diagram for $(B, R)$.

2. Find all maximal elements.

3. Partition $B$ into the smallest possible number of antichains.
15 (10pts) Prove by induction: \( \forall n \geq 1, \sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} = \frac{n}{3n+1} \).
16 (10pts) Let $G$ be a connected plane graph with $n$ vertices, $m$ edges, and $r$ regions. Prove by induction:

$$n - m + r = 2.$$ 

*Hint:* You may use the fact that a tree with $n \geq 1$ vertices has $n - 1$ edges.
17 (10pts) \( \forall n \geq 0, \)

\[
1 + 5 + 9 + \ldots + (4n + 1) = an^2 + bn + c.
\]

1. Find \( a, \ b, \) and \( c. \)

2. Prove your result by induction.

*Hint:* To find \( a, \ b, \) and \( c, \) get 3 equations in 3 unknowns.
18 (18pts) Find, if possible, a **simple connected** graph with the following properties. If not possible, explain why not.

1. 9 vertices and 13 edges that is eulerian.

2. 13 vertices, with 11 vertices of degree 4.

3. 5 vertices and eulerian, but not hamiltonian.

4. 10 vertices and 13 edges, whose vertices have degrees (4, 4, 3, 3, 3, 2, 2, 2).

5. planar, with 4 regions and 7 edges.

6. not planar, with 6 vertices and 9 edges.