6 Relations

Let $R$ be a relation on a set $A$, i.e., a subset of $A \times A$.

Notation: $xRy$ iff $(x, y) \in R \subseteq A \times A$.

Recall: A relation need not be a function.

Example: The relation $R_1 = \{(x, y) \in R \times R \mid x^2 + y^2 = 1\}$ is not a function.

Some definitions

1. $R$ is reflexive iff $xRx \ \forall x \in A$.
2. $R$ is symmetric iff $xRy \Rightarrow yRx \ \forall x, y \in A$.
3. $R$ is antisymmetric iff $xRy \land yRx \Rightarrow x = y \ \forall x, y \in A$.
4. $R$ is transitive iff $xRy \land yRz \Rightarrow xRz \ \forall x, y, z \in A$.

We now give a number of examples.

1. $R = “<“$ on $Z$. \hspace{1cm} \overline{R}, \overline{S}, A, T.
2. $R = “\leq“$ on $Z$. \hspace{1cm} R, \overline{S}, A, T.
3. $R = “=“$ on $Z$. \hspace{1cm} R, S, A, T.
4. \( R, S, \overline{A}, T. \)

5. \( R, \overline{S}, A, T. \)

6. \( R, \overline{S}, A, T. \)

7. \( R, S, \overline{A}, T. \)

8. \( R, S, \overline{A}, T. \)

9. \( R, S, A, T. \)

10. \( \overline{R}, S, \overline{A}, T. \)
Note:

1. In a reflexive relation there is a loop at each vertex.

2. In a symmetric relation, there are either 2 arcs or no arcs between any two distinct nodes.

3. In an antisymmetric relation there is either 1 arc or no arcs between any two distinct nodes.

Definition: If $R$ is a relation from $X$ to $Y$, then the inverse of $R$ is $R^{-1} = \{(y, x) \mid (x, y) \in R\}$.

Examples:

$R:$

$R^{-1}:$

$R:$

$R^{-1}:$

$R:$

$R^{-1}:$

Theorem 6.1 $R$ is symmetric iff $R = R^{-1}$.

Theorem 6.2 The reflexive, symmetric, antisymmetric and transitive properties of relations are preserved by the inverse, i.e., if $R$ has such a property then so does $R^{-1}$.
Equivalence Relations

**Definition:** A relation $R \subseteq A \times A$ is an equivalence relation if it is reflexive, symmetric and transitive.

**Examples:**

1. "$=\"$ on $\mathbb{Z}$.

2. The universal relation $U_A = A \times A$, i.e., the relation consisting of all elements of $A \times A$.

3. Let $A$ be the set of all triangles in the plane. Then $T_1 RT_2$ iff $T_1$ and $T_2$ are similar triangles.

4. Let $A$ be the set of all points in the plane. Then $p_1 Rp_2$ iff the distance from $p_1$ to the origin equals the distance from $p_2$ to the origin.

5. $A = \mathbb{Z}, m \in \mathbb{Z}, m > 0$. $aR_m b$ iff $m \mid a - b$, i.e., $\exists c \in \mathbb{Z}$ such that $m \cdot c = a - b$.
   
   (a) $R_m$ is reflexive since $m \mid a - a$.
   
   (b) $R_m$ is symmetric since $m \mid a - b \Rightarrow m \mid b - a$.
   
   (c) $R_m$ is transitive since if $m \mid a - b$ and $m \mid b - c$, then $m \mid (a - b) + (b - c)$ or $m \mid a - c$.

   **Notation:** If $m \mid a - b$ we say $a$ is congruent to $b$ mod $m$, or $a \equiv b \pmod{m}$.
6. Let \( f : A \to B \). Then \( R_f \) given by \( a_1 R_f a_2 \) iff \( f(a_1) = f(a_2) \) is an equivalence relation.

7. 

\textit{Definition:} Let \( R \) be an equivalence relation on \( A \) and \( b \in A \). Then \([b] = \{x \in A \mid xRb\}\) is the \textbf{equivalence class} generated by \( b \).

\textit{Example:} Let \( A = \mathbb{Z} \) and consider \( aR_3b \). Thus \( a \) and \( b \) are related iff \( 3 \mid a - b \). Then

\[ [0] = \{\ldots - 6, -3, 0, 3, 6, \ldots \} \]
\[ [1] = \{\ldots - 5, -2, 1, 4, 7, \ldots \} \]
\[ [2] = \{\ldots - 4, -1, 2, 5, 8, \ldots \} \]

\textit{Example:} In (7) above, \([1] = [2] = \{1, 2\}\) and \([3] = [4] = \{3, 4\}\).

\textit{Definition:} Let \( A = \bigcup_{\alpha \in \Lambda} A_\alpha \), where each \( A_\alpha \neq \emptyset \) and the \( A_\alpha \)'s are pairwise disjoint. Then \( \{A_\alpha \mid \alpha \in \Lambda\} \) is a \textbf{partition} of \( A \).

\textit{Example:} \( A_1, A_2, \ldots, A_7 \) is a partition of \( A \).
\textit{Note:} A partition of a set defines an equivalence relation in a very natural way.

\textit{Definition:} Let $P$ be a partition of a set $A$. Then the equivalence relation $R(P)$ associated with $P$ is given by: $aR(P)b$ iff $a$ and $b$ are in the same set in $P$.

\textit{Note:} $R(P)$ is clearly an equivalence relation.

\textit{Example:} The sets $A_1, A_2, A_3$ partition $\mathbb{Z}$.

\begin{align*}
A_1 &= \{ \ldots -6, -3, 0, 3, 6, \ldots \} \\
A_2 &= \{ \ldots -5, -2, 1, 4, 7, \ldots \} \\
A_3 &= \{ \ldots -4, -1, 2, 5, 8, \ldots \}
\end{align*}

Thus $1R(P)7$ and $-4R(P)8$.

We now wish to show that each equivalence relation on a set $A$ defines a partition in a natural way.
Recall \([b] = \{x \in A \mid xRb\}\).

**Theorem 6.3** Let \(R\) be an equivalence relation on a set \(A\). Then

1. \(b \in [b] \ \forall b \in A\).
2. \(\forall a, b \in A, [a] = [b] \Leftrightarrow aRb\).
3. \(\forall a, b \in A, \text{ either } [a] = [b] \text{ or } [a] \cap [b] = \emptyset\).

**Note:** Let \(\textbf{P}\) be the set of all partitions on a set \(A\) and \(\textbf{E}\) be the set of all equivalence relations on \(A\). There is a \(1 - 1\) onto function \(f : \textbf{P} \rightarrow \textbf{E}\) given by \(f(P) = R(P)\). The function is clearly \(1 - 1\). To see that it is onto, start with any equivalence relation \(E^*\) on \(A\). The partition that “maps” to it is the one obtained from the above Theorem, i.e., the partition obtained from the distinct equivalence classes of the elements of \(A\).

**Example:** Let’s define \(Q\).

First, let \(F = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}\). Define \(R \subseteq FxF\) by \(a/b R c/d\) iff \(ad = bc\).

Hence \(2/3 R 6/9\). Also, \([1/2] = \{1/2, 2/4, 3/6, 4/8, \ldots\}\).

Thus each rational is an equivalence class in \(F\) under \(R\).
Posets

Let $R$ be a relation on a set $A$. Then $R$ is a **partial ordering** on $A$ if $R$ is

1. reflexive
2. antisymmetric
3. transitive

**Examples:**

1. $R$: “$\leq$” on $\mathbb{Z}$.
2. $R$: “$\subseteq$” on $\mathcal{P}(A)$, the power set of $A$.
3. $R$: “divides” on $\mathbb{Z}^+$.  
   (a) $a | a$.
   (b) $a | b$ and $b | a \Rightarrow a = b$.
   (c) $a | b$ and $b | c \Rightarrow a | c$.
4. Let $\Sigma$ be an alphabet with a partial ordering. Then the “lexicographic” (alphabetical) ordering $R$ on $\Sigma^*$ is a partial ordering (see definition on p.636).

**Example:** $\Sigma = \{a, b\}$.

- $aab R aabab$
- $baa R bab$
- $e R baab$
- $aabb R b$
**Definition:** If $R$ is a partial ordering on $A$ we call $(A, R)$ a partially ordered set or **poset**.

**Notation:** When the relation is a partial ordering, we often use $a \leq b$ instead of $aRb$.

**Definition:** Suppose $(A, R)$ is a poset. Elements $a$ and $b$ of $A$ are said to be **comparable** if, and only if, either $aRb$ or $bRa$. Otherwise they are noncomparable.

**Definition:** Let $R$ be a partial order relation on a set $A$. If any two elements $a$ and $b$ in $A$ are comparable, then $R$ is a **total order** relation on $A$. 
Examples:

1. \( R : \leq \) on \( \mathbb{Z} \) is a total order.

2. \( R : \subseteq \) on \( P(A) \) is not a total order if \( A \) has more than 1 element.

3. \( R : \) “divides” is not a total order on \( \mathbb{Z}^+ \), e.g., 3 does not divide 5 and 5 does not divide 3.

4. \( R : \) “lexicographic” (alphabetic) ordering on \( \Sigma^* \), where \( \Sigma \) is an alphabet with a partial ordering, is a total order.

(See definition on p. 636).

Definition: Let \((A, R)\) be a poset. A subset \( B \) of \( A \) is called a chain if, and only if, each pair of elements in \( B \) is comparable. The length of a chain is the number of elements in the chain.

Note: The book has a different definition of length.

Example: The set \( P(\{a, b, c\}) \) is partially ordered with respect to subset inclusion. The set \( S = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\} \) is a chain of length 4 in \( P(\{a, b, c\}) \).
Hasse Diagrams

Let $A = \{0, 1\}$ and consider the poset $(P(A), \subseteq)$.

Properties of Hasse Diagrams

- arrows are omitted - edges are directed upward
- self loops are omitted
- edges implied by transitivity are omitted

More examples:
**Definition:** Let \((A, \leq)\) be a poset. An element \(a \in A\) is a **maximal** element if there does not exist \(b \in A\) such that \(b \neq a\) and \(a \leq b\).

**Note:** minimal element is defined similarly.

In the examples above

- \(a\) is a maximal element
- \(g\) is a minimal element
- \(1, 2, 7\) are maximal elements
- \(8, 9, 10\) are minimal elements

**Definition:** A subset of a poset \((A, R)\) is an **antichain** if no two distinct elements of the subset are related.

**Example:** \(\{c, f, e\}\)

**Theorem 6.4** Let \((A, \leq)\) be a poset. If \(n\) is the length of a longest chain in \((A, \leq)\), then \(A\) can be partitioned into \(n\) disjoint antichains.

**Proof:** Later, by induction.
Example: \( A = \{2, 3, 4, 6, 8, 12, 24, 30, 33, 60, 90, 120\} \).

\( R \): “divides” on \( A \).
Closure Operations on Relations

Example: Suppose we define a relation $R$ on a set $A$ of cities as follows: $aRb$ iff there is a direct communication link from city $a$ to city $b$ for transmission of messages.

Problem: Find a relation that describes how messages can be transmitted from one city to another, either through a direct communication link, or through any number of intermediate cities.

Definition: Let $R$ be a relation on a set $A$. The transitive closure of $R$ is a relation $R^t$ such that

1. $R^t$ is transitive
2. $R \subseteq R^t$
3. If $R_1$ is transitive and $R \subseteq R_1$, then $R^t \subseteq R_1$.

Note: The transitive closure is unique. It can be found by noting that $(x, y) \in R^t$ iff there is a “directed path” from $x$ to $y$ in the graphical representation of $R$.

Note: The reflexive and symmetric closures are defined in an analogous way.

Example:

$R$: 

$R^t$: 

89
Example:

\[ R: \quad R': \]

\[ R: \quad R^s: \]

**Theorem 6.5** Let \( \{ S_\alpha \mid \alpha \in \Lambda \} \) be the set of all transitive relations containing a relation \( R \). Then \( R' = \bigcap_{\alpha \in \Lambda} S_\alpha \).

Thus the transitive closure of a relation \( R \) is the “smallest” transitive relation containing \( R \). It is obtained by adding the least number of ordered pairs to ensure transitivity.

*Note:* A similar theorem holds for reflexive and symmetric closures.
Composition of Relations

**Definition:** Let $R_1$ be a relation from $A$ to $B$ and $R_2$ be a relation from $B$ to $C$. The **composition** of $R_1$ and $R_2$ is a relation from $A$ to $C$ given by

$$R_1R_2 = \{(a,c) \mid a \in A, c \in C \land \exists b \in B \text{ such that } [(a,b) \in R_1 \land (b,c) \in R_2]\}.$$ 

**Example:**

**Note:** In general, $R_1R_2 \neq R_2R_1$.

In fact, if $R_1$ is a relation from $A$ to $B$ and $R_2$ is a relation from $B$ to $C$, then $R_2R_1$ is not defined.

**Example:** Let $A = \{0,1,2,3\}$ and consider $R_1$ and $R_2$ on $A$. 

91
Theorem 6.6 Let $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$ and $R_3 \subseteq C \times D$. Then $(R_1 R_2) R_3 = R_1 (R_2 R_3)$, i.e., the composition of relations is associative.

Proof: (⊆) Let $(a, d) \in (R_1 R_2) R_3$. Then $\exists c \in C$ such that $(a, c) \in R_1 R_2$ and $(c, d) \in R_3$. Since $(a, c) \in R_1 R_2 \exists b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$. Now $(b, c) \in R_2$ and $(c, d) \in R_3 \Rightarrow (b, d) \in R_2 R_3$. But now $(a, b) \in R_1 \Rightarrow (a, d) \in R_1 (R_2 R_3)$.

⊇ Similar. □

Definition: Let $R$ be a binary relation on a set $A$. Then for all integers $n \geq 0$, $R^n$ is defined as follows:

1. $R^0 = \{(x, x) \mid x \in A\}$.
2. $R^{n+1} = R^n R$. 

92
Example:

• \( R^0:\)

• \( R^1 = R:\)

• \( R^2 = R^1 R:\)

• \( R^3 = R^2 R:\)

• \( R^4 = R^3 R:\)

Note: In this example, \( R^4 = R^2.\)
Theorem 6.7 Let $|A| = n$ and $R \subseteq AxA$. Then $\exists s, t$, $0 \leq s < t \leq 2^{n^2}$, such that $R^s = R^t$.

Proof: First note that $AxA$ has $n^2$ elements. Hence there are $2^{n^2}$ distinct relations on $A$. By the pigeonhole principle, at least two of them are equal. $\square$