QUANTILE-BASED DEVIATION MEASURES

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Motivation

- Spectral representation of Deviation Measures
- SSD-consistency of Deviation Measures
- Rao-Blackwell Theorem for General Deviation Measures
- Chebyshev's Theorem with General Deviation Measures

General Deviation Measures

(Rocakfellar et.al., 2002)

- **(D1) – insensitivity to constant shift**
  $\mathcal{D}(X + C) = \mathcal{D}(X)$ *for all* X *and constants* C
- **(D2) – positive homogeneity**
  $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ *for all* X *and all* $\lambda > 0$
- **(D3) – subadditivity**
  $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ *for all* X *and* Y
- **(D4) – nonnegativity**
  $\mathcal{D}(X) \geq 0$ *(equality for constant* X*)*

$\mathcal{D}(X)$ is law-invariant if $X \sim Y \Rightarrow \mathcal{D}(X) = \mathcal{D}(Y)$

Examples of Deviation Measures

- **Standard Deviation**

$$\sigma(X) = \left(E[X - EX]^2\right)^{\frac{1}{2}}$$

- **Standard Semideviations**

$$\sigma_+(X) = (E[\max\{X - EX, 0\}^2])^{1/2}$$

$$\sigma_-(X) = (E[\max\{EX - X, 0\}^2])^{1/2}$$

- **Mean Absolute Deviation**

$$MAD(X) = E|X - EX|$$

- **Deviation measure from range**

$$EX - \inf X$$

Connection with Coherent Risk Measures

$$\mathcal{R}(X) = \mathcal{D}(X) - EX, \quad \mathcal{D}(X) = \mathcal{R}(X - EX)$$

$\Rightarrow \mathcal{D}$ **is lower range dominated deviation measure:**

$$\mathcal{D}(X) \leq EX - \inf X$$

Quantile

$$q_X(\alpha) = \inf\{z | P[x \leq z] > \alpha\}$$

Conditional Value-at-Risk (CVaR)

for $\alpha \in [0,1]$ $\mathrm{CVaR}_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha q_X(p)dp$

CVaR deviation

for $\alpha \in [0,1]$ $\mathrm{CVaR}^\Delta_\alpha(X) = EX - \frac{1}{\alpha}\int_0^\alpha q_X(p)dp$

Kusuoka representation of Risk Measures

Coherent risk measure on atomless Ω (Kusuoka, 2001)

$$\mathcal{R}(X) = \sup_{\mu \in \mathcal{M}} \int_0^1 \mathrm{CVaR}_\alpha(X)\, \mu(d\alpha),$$

where $\mathcal{M}$ collection of measures μ s.t. $\mu \geq 0$, $\int_0^1 \mu(d\alpha) = 1$.

$\Rightarrow$ **lower range dominated** deviation measure

$$\mathcal{D}(X) = \sup_{\mu \in \mathcal{M}} \int_0^1 \mathrm{CVaR}_\alpha^\Delta(X)\, \mu(d\alpha),$$

(Worst-Case Mixed Deviation CVaR, see Rockafellar et.al., 2002)

Note: $\sigma(X)$ – not lower range dominated.

Quantile-based Deviation Measures

$$\mathcal{D}(X) = \sup_{\varphi \in \Lambda} \int_0^1 \varphi(\alpha)\, q_X(\alpha)\, d\alpha$$

where Λ – collection of functions $\varphi(\alpha) \in \mathcal{L}^2([0,1])$ such that

- all $\varphi(\alpha) \in \Lambda$ are non-decreasing
- $\int_0^1 \varphi(\alpha)\, d\alpha = 0 \;\; \forall \varphi(\alpha) \in \Lambda$
- Λ is non-empty and $\Lambda \neq \{0\}$.

Theorem

$$\mathcal{D}(X) = \sup_{\varphi \in \Lambda} \int\limits_0^1 \mathrm{CVaR}_\alpha^\Delta(X)\, \mu(d\alpha), \quad \mu(d\alpha) = \alpha d(\varphi(\alpha)) \geq 0$$

and $\mathcal{D}(X)$ is a lower-semicontinuous deviation measure

(but not necessarily lower range dominated)

Example

- **Standard deviation**

$$\sigma(X) = \sup_{\varphi \in \Lambda_\sigma} \int_0^1 \varphi(\alpha)\, q_X(\alpha)\, d\alpha$$

$\varphi(\alpha)$ is non-decreasing and

$$\Lambda_\sigma = \left\{ \varphi \in \mathcal{L}^2([0,1]) \;\middle|\; \int_0^1 \varphi \, d\alpha = 0, \quad \int_0^1 \varphi^2 \, d\alpha \leq 1 \right\}$$

Optimal φ is attained at

$$\varphi(\alpha) = \frac{q_X(\alpha) - EX}{\sigma(X)}$$

Characterization

Theorem
Ω - atomless $\Rightarrow$ every law-invariant lower-semicontinuous deviation measure is quantile-based

Example Lower Semideviation

$$\sigma_-(X) = (E[\max\{EX - X, 0\}^2])^{1/2}$$

Ω - atomless $\Rightarrow$ For every $X \in \mathcal{L}^2(\Omega)$

$$\sigma_-(X) = \sup_{\varphi \in \Lambda} \int_0^1 \varphi(\alpha) q_X(\alpha) d\alpha$$

Ω - arbitrary: Use the same Λ.

Example: Not Quantile-Based D(X)

$\Omega = \{\omega_1, \omega_2\}$ $P\{\omega_1\} = \frac{1}{3}$, $P\{\omega_2\} = \frac{2}{3}$.

$X \leftrightarrow (x_1, x_2)$ $\quad x_1 = X(\omega_1)$, $x_2 = X(\omega_2)$

$$\mathcal{D}(X) = \begin{cases} 3(x_2 - x_1), & x_2 \geq x_1 \\ x_1 - x_2, & x_1 \geq x_2 \end{cases}$$

$\mathcal{D}(X)$ - law-invariant and continuous.

Proposition

$\mathcal{D}(X)$ is **not** quantile-based.

Second-Order Stochastic Dominance (SSD)

$$X \succcurlyeq_{SSD} Y \iff \forall t \int_{-\infty}^{t} F_X(x)dx \le \int_{-\infty}^{t} F_Y(y)dy$$

Risk-averse investor always preferres X to Y.

Coherent risk measures:

$$X \succcurlyeq_{SSD} Y \Rightarrow \mathcal{R}(X) \le \mathcal{R}(Y)$$

(if $\mathcal{R}$ – lower semicontinuous, law-invariant, Ω – atomless)

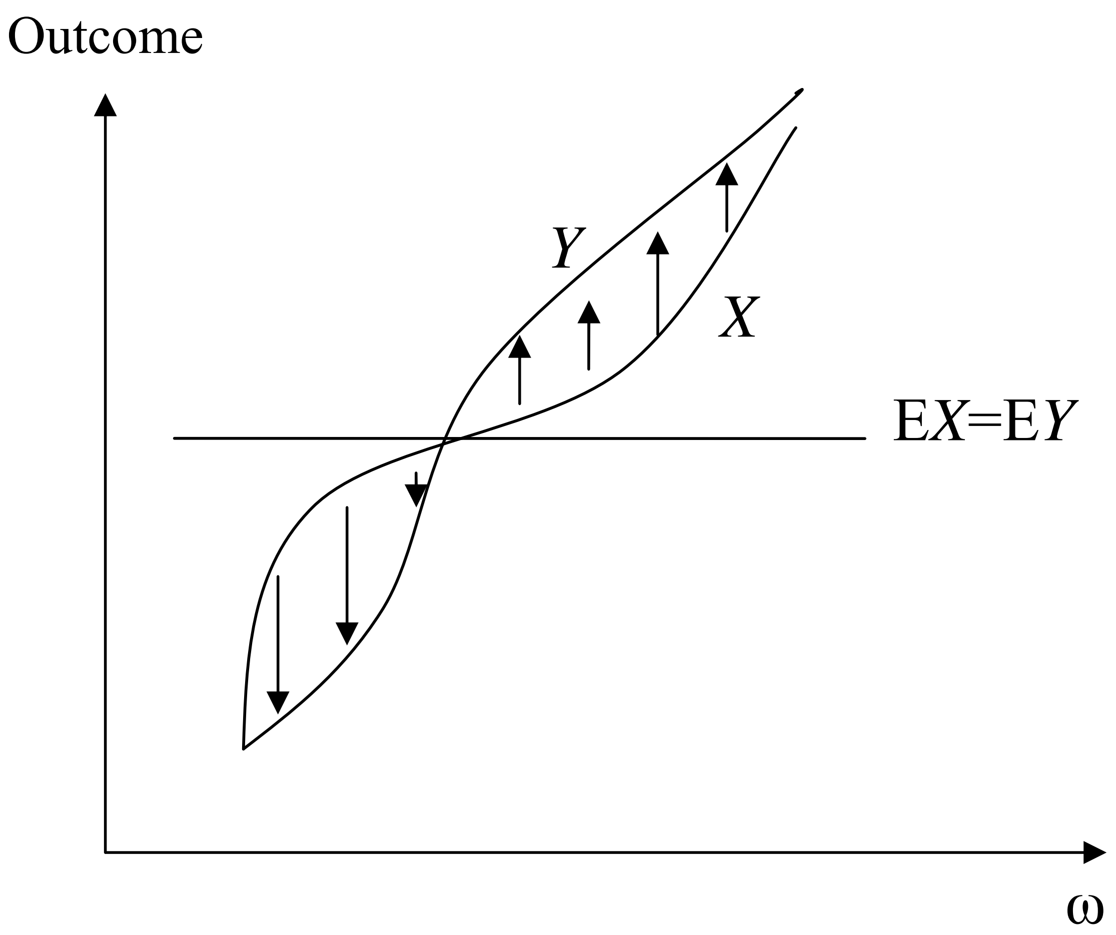
Deviation measures:

$$EX = EY,\ X \succcurlyeq_{SSD} Y \overset{?}{\Rightarrow} \mathcal{D}(X) \le \mathcal{D}(Y)$$

SSD and Quantile-Based Deviation Measures

Theorem
Let $\mathcal{D}(X)$ - lower semicontinuous deviation measure.
Then $\mathcal{D}(X)$ is quantile-based if and only if

$$EX = EY,\ X \succcurlyeq_{SSD} Y \Rightarrow \mathcal{D}(X) \leq \mathcal{D}(Y)$$



Rao-Blackwell Theorem

Classical Theorem:
Given X, Y, define $Z = E[X|Y]$.
Then $EZ = EX$ and $\sigma(Z) \leq \sigma(X)$.

$Z = E[X|Y] \succcurlyeq_2 X$

Generalized Theorem:
$\mathcal{D}$ – quantile-based $\Rightarrow$
$\forall X, Y : EZ = EX$, $\mathcal{D}(Z) \leq \mathcal{D}(X)$.

Risk Envelope

For lower semicontinuous $\mathcal{D}(X)$

$$\mathcal{D}(X) = \sup_{Q \in \mathcal{Q}} E[(1-Q)X]$$

where $\mathcal{Q}$ is

(Q1) $\mathcal{Q} \subset \mathcal{L}^2(\Omega)$ – closed, convex, $1 \in \mathcal{Q}$

(Q2) $EQ = 1 \; \forall Q \in \mathcal{Q}$

(Q3) $\forall X \neq C \; \exists Q \in \mathcal{Q}$ s.t. $E[QX] < EX$.

$$\mathcal{Q} = \left\{ Q \in \mathcal{L}^2 \,\middle|\, E[(1-Q)X] \leq \mathcal{D}(X) \text{ for all } X, \; EQ = 1 \right\}$$

$\mathcal{D}$ is lower range dominated $\iff Q \geq 0 \quad \forall Q \in \mathcal{Q}$

Link to Quantile-Based Deviation Measures

$$\forall \mathcal{D}: \ \mathcal{D}(X) = \sup_{Q \in \mathcal{Q}} E[(1-Q)X]$$

Proposition

$\mathcal{D}(X)$ – quantile-based if and only if

$$D(X) = \sup_{Q \in \mathcal{Q}} \int_0^1 q_{1-Q}(\alpha) q_X(\alpha) d\alpha$$

Namely,

$$D(X) = \sup_{\varphi \in \Lambda} \int_0^1 \varphi(\alpha) q_X(\alpha) d\alpha$$

with

$$\Lambda = \{\varphi \,|\, \varphi(\alpha) = q_{1-Q}(\alpha), \ Q \in \mathcal{Q}\}$$

Chebyshev Theorem for Deviation Measure

Classical Theorem:

$$P\{|X - EX| \geq a\} \leq \frac{\sigma(X)^2}{a^2}$$

Problem: For every $D(X)$ find $f_D(D(X), a)$ such that

$$P\{|X - EX| \geq a\} \leq f_D(D(X), a)$$

Requirements:

- form of $f_D(.)$ should not depend on X and Ω
- $f_D(.)$ should be smallest possible bound.

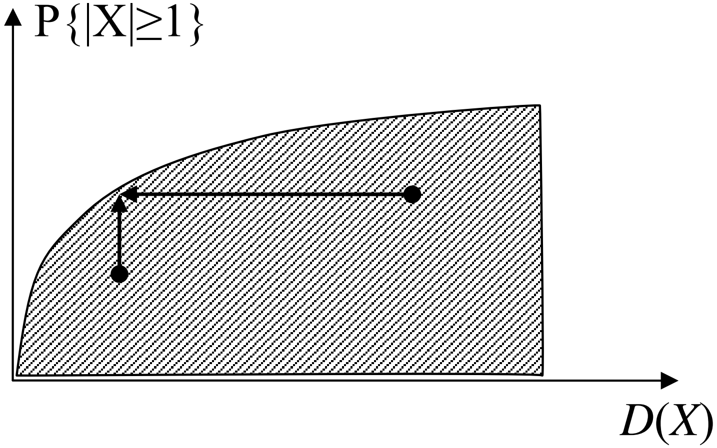
Chebyshev Theorem: Problem Restatement

Deviation measures axioms:

- $D(X + C) = D(X)$
- $D(\lambda X) = \lambda D(X)$

$$P\{|X| \geq 1\} \leq f_D(D(X)) \quad \forall X \in \mathcal{L}^2(\Omega), EX = 0$$

$$f_D(d) = \max_X P\{|X| \geq 1\}$$
$$\text{s.t. } D(X) \leq d$$



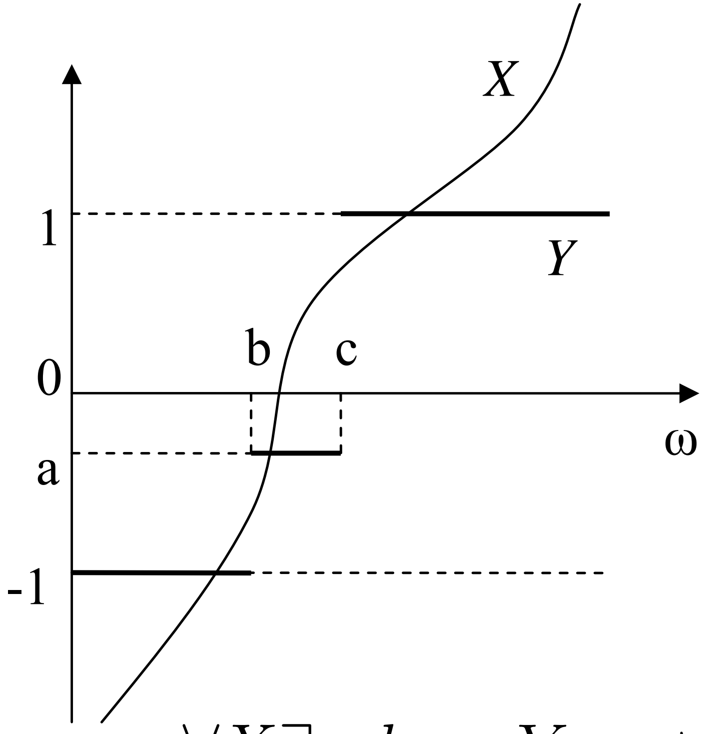
$$u_D(\beta) = \min_X D(X)$$
$$\text{s.t. } P\{|X| \geq 1\} \geq \beta$$

$$f_D(d) = \sup\{\beta \,|\, u_D(\beta) \leq d\}$$

Chebyshev Theorem: Solution

$$u_D(\beta) = \min_X D(X)$$

$$\text{s.t. } EX = 0,\ P\{|X| \geq 1\} \geq \beta$$



$$Y_{a,b,c} = \begin{cases} -1, & p_1 = b \\ a, & p_2 = c - b \\ 1, & p_3 = 1 - c \end{cases}$$

$$EY_{a,b,c} = 0$$

$$P\{|Y_{a,b,c}| \geq 1\} \geq \beta$$

$$\forall X \exists a, b, c : Y_{a,b,c} \succcurlyeq_{SSD} X \Rightarrow \mathcal{D}(X) \geq \mathcal{D}(Y_{a,b,c})$$

$$u_D(\beta) = \min_{a,b,c} \mathcal{D}(Y_{a,b,c})$$

Examples of Chebyshev Inequalities

Standard deviation

- Symmetric

$$P\{|X - EX| \geq a\} \leq \frac{\sigma(X)^2}{a^2}$$

- Left tail

$$P\{X - EX \leq -a\} \leq \frac{\sigma(X)^2}{a^2 + \sigma(X)^2}$$

Lower semi-deviation

- Left tail

$$P\{X - EX \leq -a\} \leq \frac{\sigma_-(X)^2}{a^2}$$

Examples of Chebyshev Inequalities

Mean-absolute deviation

- Symmetric

$$P\{|X - EX| \geq a\} \leq \frac{MAD(X)}{a}$$

- Left tail

$$P\{X - EX \leq -a\} \leq \frac{MAD(X)}{2a}$$

CVaR

- Left tail

$$P\{X - EX \leq -a\} \leq \frac{\alpha \mathrm{CVaR}_{\alpha}^{\Delta}(X)}{a + \alpha(\mathrm{CVaR}_{\alpha}^{\Delta}(X) - a)}$$

Concluding Remarks

Quantile-based deviation measures:

- **Include all well-known deviation measures**
- **Several equivalent representations**
- **SSD-consistent**
- **Rao-Blackwell theorem**
- **Chebyshev inequalities**