Stochastic dynamic optimization with multivariate stochastic dominance constraints

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References:

Optimization with stochastic dominance constraints,
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Stochastic optimization with nonlinear dominance constraints,
Mathematical Programming 99 (2004) 329–350

Stochastic dynamic optimization with discounted stochastic dominance constraints,
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available from `www.rusz.rutgers.edu`

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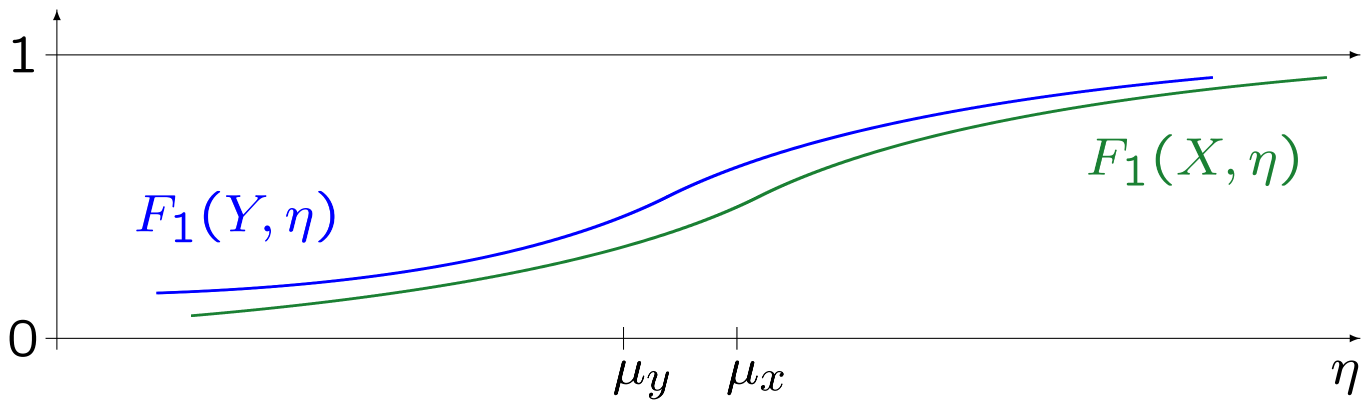
First Order Dominance

Notation: $F_1(X,\eta) = \mathbb{P}\{X \le \eta\}$

$$X \succeq_{(1)} Y \quad \Leftrightarrow \quad F_1(X,\eta) \le F_1(Y,\eta) \quad \text{for all } \eta \in \mathbb{R}$$



Relation to utility functions:

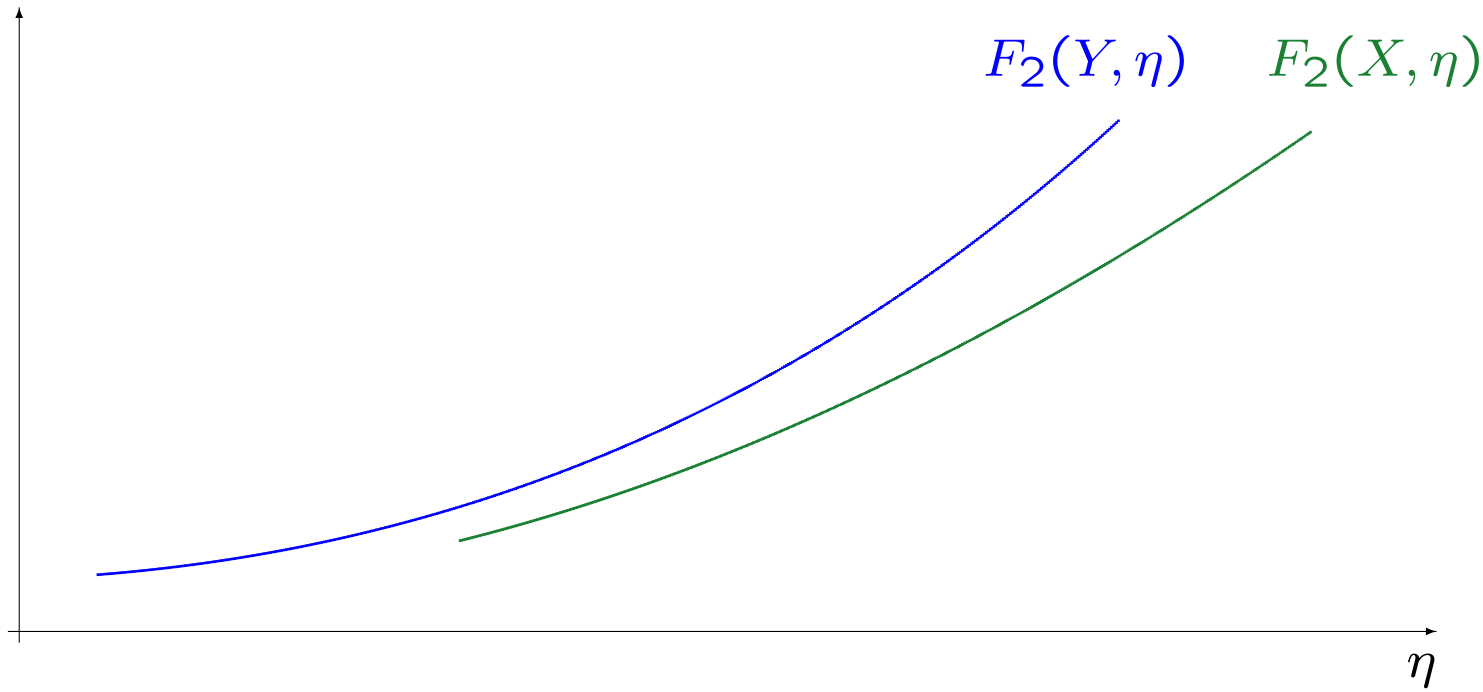
$$X \succeq_{(1)} Y \;\Leftrightarrow\; \mathbb{E}\,u(X) \ge \mathbb{E}\,u(Y) \quad \forall \textit{ nondecreasing } u(\cdot)$$

Quirk and Saposnik (1962)

Second Order Dominance

Notation: $F_2(X,\eta) = \int_{-\infty}^{\eta} F_1(X,\xi)\,d\xi = \mathbb{E}(\eta - X)_+$ for $\eta \in \mathbb{R}$

$$X \succeq_{(2)} Y \quad \Leftrightarrow \quad F_2(X,\eta) \le F_2(Y,\eta) \quad \text{for all } \eta \in \mathbb{R}$$



Relation to utility functions:

$$X \succeq_{(2)} Y \;\Leftrightarrow\; \mathbb{E}\,u(X) \ge \mathbb{E}\,u(Y) \quad \forall \text{ nondecreasing concave } u(\cdot)$$

Hadar and Russell (1969)

Dominance Constrained Optimization Problems

$$\max \ \mathbb{E}\Big[H(z)\Big]$$
$$\text{subject to } G(z) \succeq_{(2)} Y$$
$$z \in Z_0$$

Z_0 - convex closed subset of a Banach space $\mathscr{Z}$
G and H - continuous concave operators from $\mathscr{Z}$ to $\mathscr{L}_1(\Omega, \mathscr{F}, P)$
Y - benchmark outcome in $\mathscr{L}_1(\Omega, \mathscr{F}, P)$

Convenient re-formulation of the dominance constraint:

$$F_2(G(z); \eta) \le F_2(Y; \eta) \quad \text{for all } \eta \in [a, b]$$

How to formulate the order $\succeq_{(2)}$ if the outcome and the benchmark are multivariate?

$$G(z) = \Big(G_1(z), G_2(z), \ldots, G_{T+1}(z)\Big)$$
$$Y = (Y_1, Y_2, \ldots, Y_{T+1})$$

Important case: dynamic with discrete time

Multivariate Stochastic Orders

Consider $X = (X_1, \dots, X_{T+1})$ and $Y = (Y_1, \dots, Y_{T+1})$ in $\mathscr{L}_1^{T+1}(\Omega, \mathscr{F}, P)$.

Increasing Convex Order

$$X \succeq_{(\text{icx})} Y \;\Leftrightarrow\; \mathbb{E}\, u(X) \geq \mathbb{E}\, u(Y) \quad \forall\, u \in \mathscr{U}$$

Generator $\mathscr{U}$ - all concave nondecreasing functions $u : \mathbb{R}^{T+1} \to \mathbb{R}$

Hard to treat analytically, the generator is too rich

Coordinate Order

$$X \succeq_{(2)}^{\text{sep}} Y \;\Leftrightarrow\; X_t \succeq_{(2)} Y_t,\; t = 1, \dots, T+1$$

Generator $\mathscr{U}$ - all functions $u(X) = \sum_{t=1}^{T+1} u_t(X_t)$ with concave nondecreasing $u_t : \mathbb{R} \to \mathbb{R}$. Our earlier analysis covers this case.

Ignores temporal structure and dependency

Our idea is to define the mutlivariate order via a family of univariate orders

Dynamic Stochastic Dominance

Consider $X = (X_1, \ldots, X_{T+1})$ and $Y = (Y_1, \ldots, Y_{T+1})$ in $\mathscr{L}_1^{T+1}(\Omega, \mathscr{F}, P)$.

New idea: Consider linear scalarization by comparing

$$\langle \varrho, X \rangle = \sum_{t=1}^{T+1} \varrho_t X_t \quad \text{and} \quad \langle \varrho, Y \rangle = \sum_{t=1}^{T+1} \varrho_t Y_t$$

for all

$$\varrho \in \mathscr{D} \subseteq \left\{ \varrho \in \mathbb{R}^{T+1} : 1 \geq \varrho_1 \geq \varrho_2 \geq \cdots \geq \varrho_{T+1} \geq 0 \right\}$$

Discounted Order: $X \succeq_{(2)}^{\text{dis}} Y$ if $\langle \varrho, X \rangle \succeq_{(2)} \langle \varrho, Y \rangle$ for all $\varrho \in \mathscr{D}$

The order $\succeq_{(2)}^{\text{dis}}$ neither implies nor is implied by the coordinate order

Special case:

Finite set $\mathscr{D}$ consisting of $T+1$ elements ϱ^k, $k = 1, \ldots, T+1$, with

$$\varrho_t^k = \begin{cases} 1 & \text{if } \; t \le k, \\ 0 & \text{if } \; t > k. \end{cases}$$

Then $X \succeq_{(2)}^{\text{dis}} Y$ is equivalent to the dominance of partial sums

$$\sum_{t=1}^{k} X_t \succeq_{(2)} \sum_{t=1}^{k} Y_t, \quad k = 1, \ldots, T+1.$$

By specifying the set $\mathscr{D}$ we can obtain other interesting cases.

Similar ideas can be employed in the multivariate case, without the dynamic structure

The Generator of $\succeq_{(2)}^{\text{dis}}$

Notation: $\mathscr{M}_+(S)$ - the space of nonnegative regular measures on $S \subset \mathbb{R}^m$

For every $\lambda \in \mathscr{M}_+(\mathbb{R} \times \mathscr{D})$ we define a concave nondecreasing function $\varphi_\lambda : \mathbb{R}^{T+1} \to \mathbb{R}$:

$$\varphi_\lambda(x) = -\int_{\mathbb{R}\times\mathscr{D}} \max(0, \eta - \langle \varrho, x \rangle)\ \lambda(d(\eta, \varrho)).$$

Theorem The class of functions

$$\Phi = \{\varphi_\lambda : \lambda \in \mathscr{M}_+(\mathbb{R} \times \mathscr{D})\}$$

is a generator of the order $\succeq_{(2)}^{\text{dis}}$:

$$X \succeq_{(2)}^{\text{dis}} Y \quad \Leftrightarrow \quad \mathbb{E}[\varphi(X)] \geq \mathbb{E}[\varphi(Y)] \quad \text{for all } \varphi \in \Phi$$

Obtained by integrating w.r.t. marginal and conditional measures of λ

Dynamic Optimization Problem

$$\begin{aligned}
\text{max} \ & \sum_{t=1}^{T} \mathbb{E} G_t(s_t, v_t) + \mathbb{E} G_{T+1}(s_{T+1}) \\
\text{s.t.} \ & s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t = 1, \dots, T, \\
& (G_1(s_1, v_1), \dots, G_1(s_T, v_T), G_{T+1}(s_{T+1})) \succeq_{(2)}^{\text{dis}} (Y_1, \dots, Y_T, Y_{T+1}) \\
& v_t \in V_t \text{ a.s.}, \quad t = 1, \dots, T.
\end{aligned}$$

Decision vector z: controls $v_1, \dots, v_T$, and trajectories $s_2, \dots, s_{T+1}$

$\mathscr{F}_t$ - σ-algebra generated by $\{e_1, \dots, e_{t-1}\}$
Space of controls: $\mathscr{V} = \mathscr{L}_1^{n_v}(\Omega, \mathscr{F}_1, P) \times \cdots \times \mathscr{L}_1^{n_v}(\Omega, \mathscr{F}_T, P)$.
Space of state trajectories: $\mathscr{S} = \mathscr{L}_1^{n_s}(\Omega, \mathscr{F}_2, P) \times \cdots \times \mathscr{L}_1^{n_s}(\Omega, \mathscr{F}_{T+1}, P)$.

For technical reasons we restrict the range of $\eta \in \mathbb{R}$ to an interval $[a, b]$:

$$F_2(\langle \varrho, G(s, v) \rangle; \eta) \le F_2(\langle \varrho, Y \rangle; \eta) \quad \text{for all } \varrho \in \mathscr{D} \text{ and all } \eta \in [a, b]$$

The Partial Lagrangian

Define the class of functions

$$\Phi([a,b],\mathscr{D}) = \{\varphi_\lambda : \lambda \in \mathscr{M}_+([a,b]\times\mathscr{D})\},$$

$$\varphi_\lambda(x) = -\int_{[a,b]\times\mathscr{D}} \max(0, \eta - \langle \varrho, x\rangle)\ \lambda(d(\eta,\varrho))$$

The set $\Phi([a,b],\mathscr{D})$ is a generator of the interval dominance order

We introduce the partial "Lagrangian" $L : \mathscr{S} \times \mathscr{V} \times \Phi([a,b],\mathscr{D}) \to \mathbb{R}$:

$$L(s,v,\varphi) = \mathbb{E}\Big[\sum_{t=1}^{T} G_t(s_t,v_t) + G_{T+1}(s_{T+1})$$
$$+ \Big(\varphi(G_1(s_1,v_1),\dots,G_T(s_T,v_T),G_{T+1}(s_{T+1})) - \varphi(Y_1,\dots,Y_T,Y_{T+1})\Big)\Big]$$

Non-decomposable w.r.t. $t = 1,\dots,T+1$

Auxiliary Control Problem

$$\begin{aligned}
\max \ \mathbb{E}\Big[\sum_{t=1}^{T} G_t(s_t, v_t) + G_{T+1}(s_{T+1}) \\
+ \Big(\varphi(G_1(s_1, v_1), \dots, G_T(s_T, v_T), G_{T+1}(s_{T+1})) - \varphi(Y_1, \dots, Y_T, Y_{T+1}) \Big) \Big]
\end{aligned}$$

$$\begin{aligned}
\text{s.t.} \quad & s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t = 1, \dots, T \\
& v_t \in V_t \quad \text{a.s.}, \quad t = 1, \dots, T
\end{aligned}$$

Uniform Dominance Condition: $\exists$ a feasible pair $(\tilde{s}, \tilde{v})$ such that

$$\inf_{(\eta,\varrho)\in[a,b]\times\mathscr{D}} \Big\{ F_2(\langle \varrho, Y \rangle; \eta) - F_2(\langle \varrho, G(\tilde{s}, \tilde{v}) \rangle; \eta) \Big\} > 0.$$

Theorem

Assume the uniform dominance condition. If $(\hat{s}, \hat{v})$ is an optimal solution of the original problem then there exist $\hat{\varphi} \in \Phi([a, b], \mathscr{D})$ such that $(\hat{s}, \hat{v})$ is an optimal solution of the auxiliary problem with $\varphi = \hat{\varphi}$, and

$$\mathbb{E}[\hat{\varphi}(G(\hat{s}, \hat{v}))] = \mathbb{E}[\hat{\varphi}(Y)].$$

Conversely, if for some $\hat{\varphi} \in \Phi([a, b], \mathscr{D})$ an optimal solution $(\hat{s}, \hat{v})$ of the auxiliary problem satisfies the dominance constraint, then it is optimal.

Ideas of the Derivation

1. Write the dominance constraint as an operator constraint in $\mathscr{C}([a,b],\mathscr{D})$:

$$\mathbb{E}(\eta - \langle \varrho, G(s,v)\rangle)_+ \le \mathbb{E}(\eta - \langle \varrho, Y\rangle)_+ \quad \text{for all } \varrho \in \mathscr{D} \text{ and all } \eta \in [a,b]$$

2. Associate with it a Lagrange multiplier

   - nonnegative measure $\lambda \in \mathscr{M}_+([a,b]\times\mathscr{D})$

3. Change the order of integration in the Lagrangian term

$$\int_{[a,b]\times\mathscr{D}} \mathbb{E}(\eta - \langle \varrho, X\rangle)_+ \, \lambda(d(\eta,\varrho)) = \int_\Omega \int_{[a,b]\times\mathscr{D}} (\eta - \langle \varrho, X(\omega)\rangle)_+ \, \lambda(d(\eta,\varrho)) \, P(d\omega)$$
$$= -\mathbb{E}\varphi(X)$$

with

$$\varphi(x) = -\int_{[a,b]\times\mathscr{D}} (\eta - \langle \varrho, x\rangle)_+ \, \lambda(d(\eta,\varrho))$$

Implied Random Discount

$$\begin{aligned}
\max \quad & \sum_{t=1}^{T} \mathbb{E}G_t(s_t, v_t) + \mathbb{E}G_{T+1}(s_{T+1}) \\
\text{s.t.} \quad & s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t = 1, \dots, T, \\
& (G_1(s_1, v_1), \dots, G_1(s_T, v_T), G_{T+1}(s_{T+1})) \succeq_{(2)}^{\text{dis}} (Y_1, \dots, Y_T, Y_{T+1}) \\
& v_t \in V_t \text{ a.s.}, \quad t = 1, \dots, T.
\end{aligned}$$

Theorem Assume the uniform dominance condition. If $(\hat{s}, \hat{v})$ is an optimal solution of the original control problem then there exist $\xi_t \in \mathscr{L}_\infty(\Omega, \mathscr{F}_t, P)$, $t = 1, \dots, T+1$, with

$$\xi_1 \geq \xi_2 \geq \cdots \geq \xi_T \geq \xi_{T+1} \geq 0 \quad \text{a.s.}$$

such that $(\hat{s}, \hat{v})$ is an optimal solution of the control problem

$$\begin{aligned}
\max \quad & \sum_{t=1}^{T} \mathbb{E}(1 + \xi_t) G_t(s_t, v_t) + \mathbb{E}(1 + \xi_{T+1}) G_{T+1}(s_{T+1}) \\
\text{s.t.} \quad & s_{t+1} = A_t s_t + B_t v_t + e_t, \quad t = 1, \dots, T, \\
& v_t \in V_t \quad \text{a.s.}, \quad t = 1, \dots, T.
\end{aligned}$$

Maximum Principle

Theorem Assume the uniform dominance condition. If $(\hat{s}, \hat{v})$ is an optimal solution of the original control problem, then there exist:
discount factors $\xi_t \in \mathscr{L}_\infty(\Omega, \mathscr{F}_t, P)$, $t = 1, \ldots, T+1$,

$$\xi_1 \geq \xi_2 \geq \cdots \geq \xi_T \geq \xi_{T+1} \geq 0 \quad \text{a.s.}$$

subgradients $(\sigma_t^s, \sigma_t^v) \in \mathscr{L}_q^{n_s}(\Omega, \mathscr{F}_t, P) \times \mathscr{L}_q^{n_v}(\Omega, \mathscr{F}_t, P)$ satisfying

$$(\sigma_t^s(\omega), \sigma_t^v(\omega)) \in \partial g_t(\hat{s}_t(\omega), \hat{v}_t(\omega))$$
$$\sigma_{T+1}^s(\omega) \in \partial g_{T+1}(\hat{s}_{T+1}(\omega))$$

and dual variables $\hat{y}_t \in \mathscr{L}_q^{n_s}(\Omega, \mathscr{F}_{t+1}, P)$, $t = 1, \ldots, T$, satisfying the adjoint equations

$$y_T = (1 + \xi_{T+1})\sigma_{T+1}^s$$
$$y_{t-1} = A_t' \mathbb{E}[y_t | \mathscr{F}_t] + (1 + \xi_t)\sigma_t^s, \quad t = T, \ldots, 2$$

such that for almost all $\omega \in \Omega$ the control $\hat{v}_t(\omega)$ is a solution of

$$\max_{c \in V_t} \left\langle (1 + \xi_t(\omega))\sigma_t^v(\omega) + B_t'(\omega)\mathbb{E}[y_t | \mathscr{F}_t](\omega), c \right\rangle$$