MEAN-DEVIATION ANALYSIS IN THE THEORY OF CHOICE

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Outline

- Motivation: Preference Axioms for Mean-Deviation Analysis
- Preferences: Basic Axioms and Numerical Representation
- Expected Utility and Dual Utility
- Coherent Risk Measures: Axiomatic Derivation
- Paradoxes and Dual Paradoxes
- Mean-Deviation Analysis: Axiomatic Derivation
- Concluding Remarks

Decision Making Under Uncertainty

- **Expected Utility Theory**

  von Neumann, Morgenstern (1944)

- **Dual Utility Theory**

  Yaari (1987)

- **Coherent Risk Measures**

  Artzner, Delbaen, Eber, Heath (1999)

- **Mean-Variance Analysis**

  Markowitz (1952)

  $V(EX, \sigma(X)) \quad V \uparrow$ in EX, $\quad V \downarrow$ in $\sigma(X)$

- **Mean-Deviation Analysis**

  Rockafellar, Uryasev, Zabarankin (2002)

  $V(EX, \mathcal{D}(X))$

Basic Axioms of Rational Behaviour

Theory of choice: Preference relation $\succeq$

A1 Neutrality

$F_X = F_Y \Rightarrow X \sim Y$

A2 Complete weak order

$\succeq$ *is reflexive, transitive and connected*

A3 Strict monotonicity

$X \geq Y \Rightarrow X \succeq Y$. *Moreover, for constants* $C_1 > C_2 \Rightarrow C_1 \succ C_2$.

A4 $\mathcal{L}^p$-continuity

$X_n \to X$, $Y_n \to Y$, $\forall n\, X_n \succeq Y_n \Rightarrow X \succeq Y$

A5 Finiteness

$\forall X\ \exists C_1, C_2$ *such that* $C_1 \succeq X \succeq C_2$.

Numerical representation

Theorem.

Let $\succeq$ satisfies axioms $A1 - A5$.

Then $\exists! U : \mathcal{L}^p(\Omega) \to \mathbb{R}$, such that

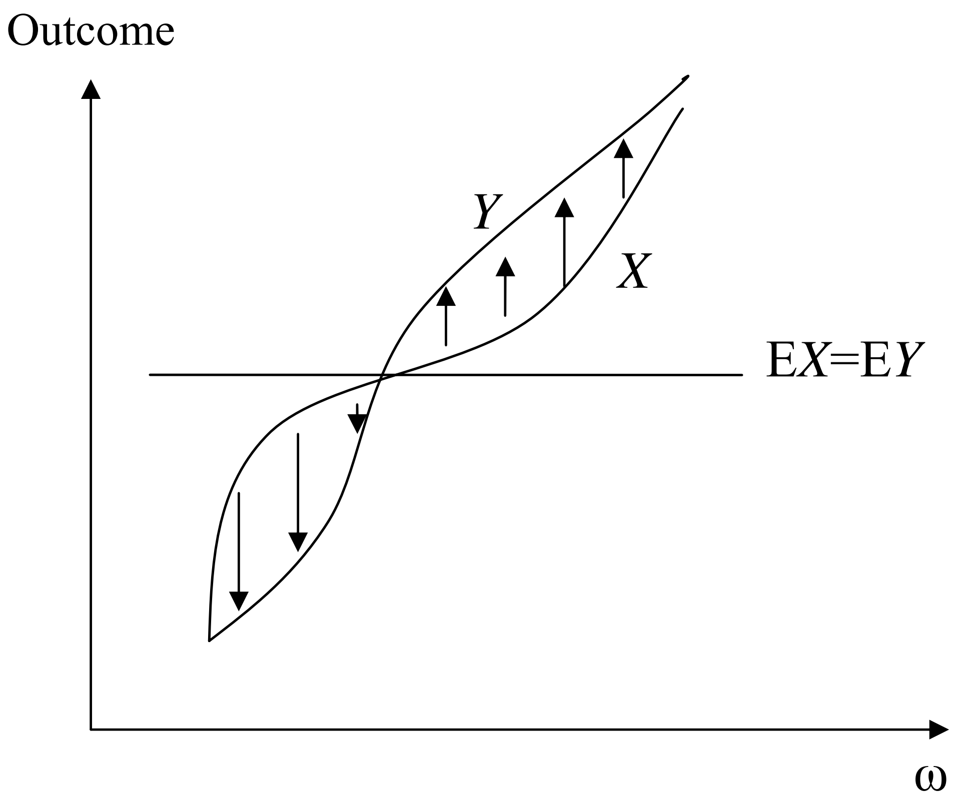
- $X \succeq Y \Leftrightarrow U(X) \geq U(Y)$.
- For every constant r. v. C we have $U(C) = C$

Moreover, $\forall X$ we have $X \sim U(X)$.

$U(X)$ is called certainty equivalent.

Second-Order Stochastic Dominance (SSD)

$$X \succcurlyeq_{SSD} Y \iff \forall t \int_{-\infty}^{t} F_X(x)dx \leq \int_{-\infty}^{t} F_Y(y)dy$$



A6 **Risk-averseness** $X \succcurlyeq_{SSD} Y, EX = EY \Rightarrow X \succeq Y$

Expected utility theory

Lottery: $X \to F_X$, $Y \to F_Y$, $Z \to F_Z = \lambda F_X + (1-\lambda)F_Y \Rightarrow$

Z is called λ-lottery of X and Y.

Notation: $Z = \lambda X \oplus (1-\lambda)Y$

A7 Independence axiom

$X \succeq Y \Leftrightarrow \lambda X \oplus (1-\lambda)Z \succeq \lambda Y \oplus (1-\lambda)Z$

Theorem.

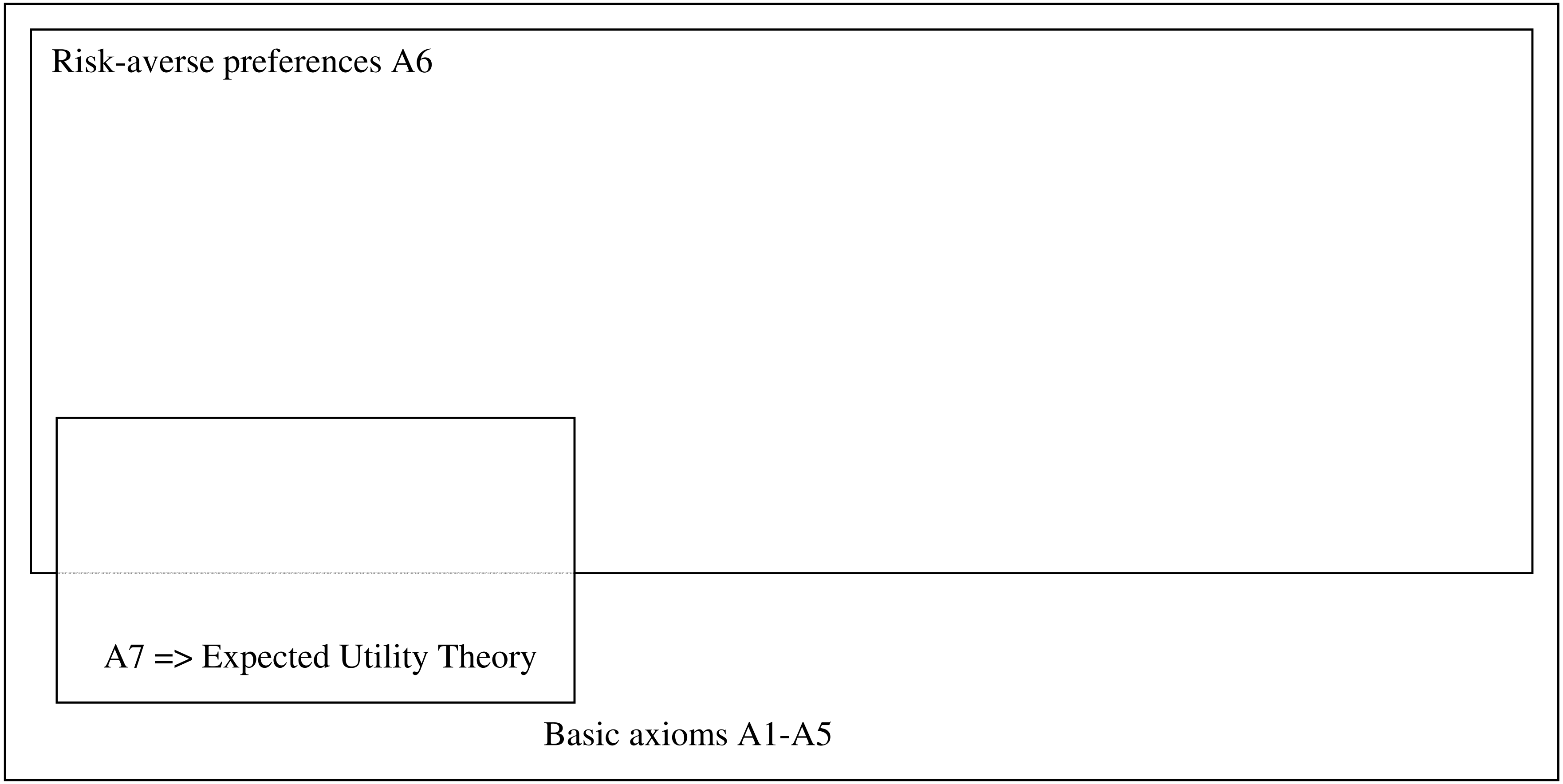
Let $\succeq$ satisfies axioms $A1 - A5$ and $A7$.

Then $U(X) = Eu(X)$ for some non-decreasing $u : R \to R$

$X \succeq Y \Leftrightarrow Eu(X) \geq Eu(Y)$

$\succeq$ satisfies also $A6 \Leftrightarrow u(.)$ is concave.

Theories Interconnection



Dual utility theory

Idea: Lottery $\rightarrow$ Mixture of comonotonic random variables.

X and Y are comonotonic $\Leftrightarrow$

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0 \;\; \forall \omega_1, \omega_2 \in \Omega$$

A8. Dual independence axiom

Let X, Y, Z be comonotonic. Then

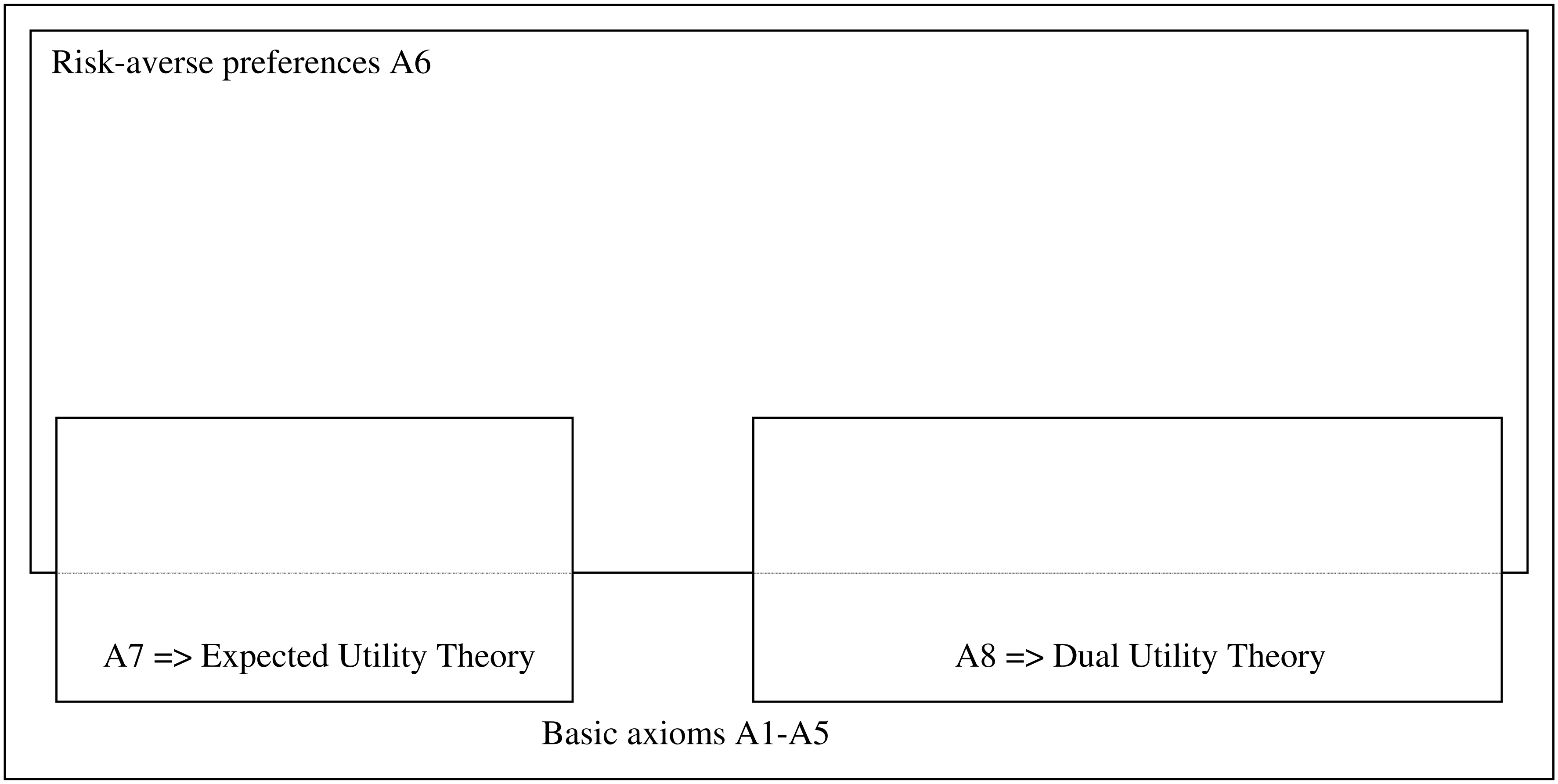
$$X \succeq Y \Leftrightarrow \lambda X + (1-\lambda)Z \succeq \lambda Y + (1-\lambda)Z$$

Theorem. Let $\succeq$ satisfies axioms $A1 - A5$ and $A8$.

Then $U(X) = \int_0^1 f(1 - F_X(t))dt$ for some $f : R \rightarrow R$

$\succeq$ satisfies also $A6 \Leftrightarrow f(.)$ is convex.

Theories Interconnection



Coherent Risk Measures

Artzner, Delbaen, Eber, and Heath

(R1) **constant translation**

$\mathcal{R}(X + C) = \mathcal{R}(X) - C$ for all X and constants C

(R2) **positive homogeneity**

$\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for all X and all $\lambda > 0$

(R3) **subadditivity**

$\mathcal{R}(X + Y) \leq \mathcal{R}(X) + \mathcal{R}(Y)$ for all X and Y

(R4) **monotonicity**

$\mathcal{R}(X) \leq \mathcal{R}(Y)$ when $X \geq Y$ (almost surely)

Expectation-Bounded Risk Measures

(R5) **strict expectation-boundedness**

$\mathcal{R}(X) > E[-X]$ for $X \not\equiv C$

R1, R2, R3, R5 — stricly expectation bounded risk measures

Comonotonic risk measures

Risk measure $\mathcal{R}(X)$ is comonotonic if

$\forall$ comonotonic X and $Y \Rightarrow \mathcal{R}(X+Y) = \mathcal{R}(X) + \mathcal{R}(Y)$

Theorem.

Let $\succeq$ satisfies axioms $A1 - A6$ and $A8$.

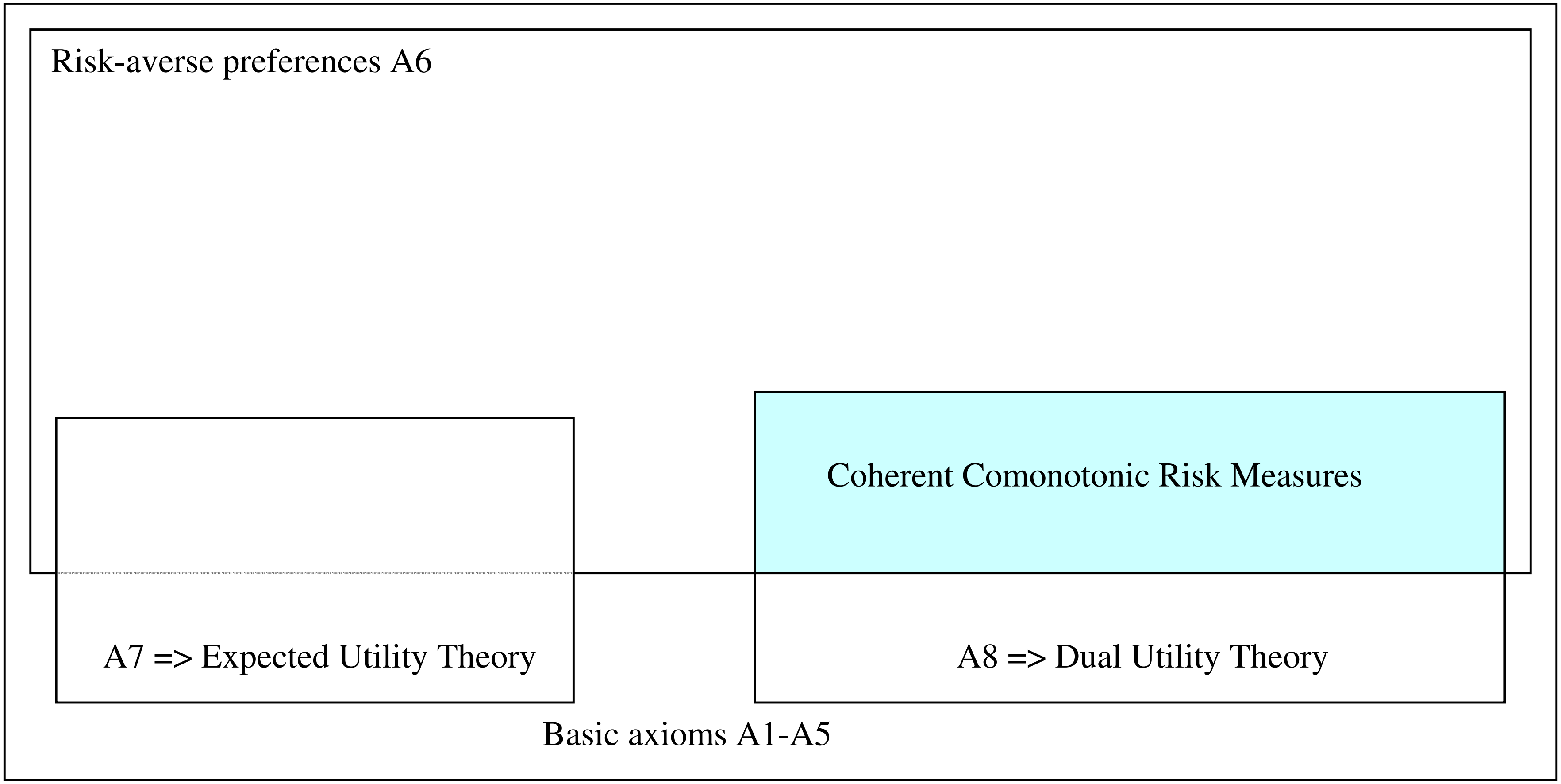
Then $U(X) = -\mathcal{R}(X)$,

$\mathcal{R}(X)$ is coherent risk measure which is

- continuous
- law-invariant
- comonotonic

$$X \succeq Y \Leftrightarrow \mathcal{R}(X) \leq \mathcal{R}(Y)$$

Theories Interconnection



Dual Independence Axiom Relaxation

Dual independence axiom $\Rightarrow$

- $\Rightarrow$ Convexity
- $\Rightarrow$ Constant absolute risk aversion
- $\Rightarrow$ Constant relative risk aversion

A9 **Convexity**

X *and* Y *comonotonic,* $X \sim Y \Rightarrow \lambda X + (1-\lambda)Y \succeq Y$

A10 **Constant absolute risk aversion**

$X \succeq Y$, C *constant* $\Rightarrow X + C \succeq Y + C$

A11 **Constant relative risk aversion**

$X \succeq Y$, $\lambda > 0 \Rightarrow \lambda X \succeq \lambda Y$

Coherent risk measures

Theorem.

Let $\succeq$ satisfies axioms $A1 - A6$ and $A9 - A11$.

Then $U(X) = -\mathcal{R}(X)$,

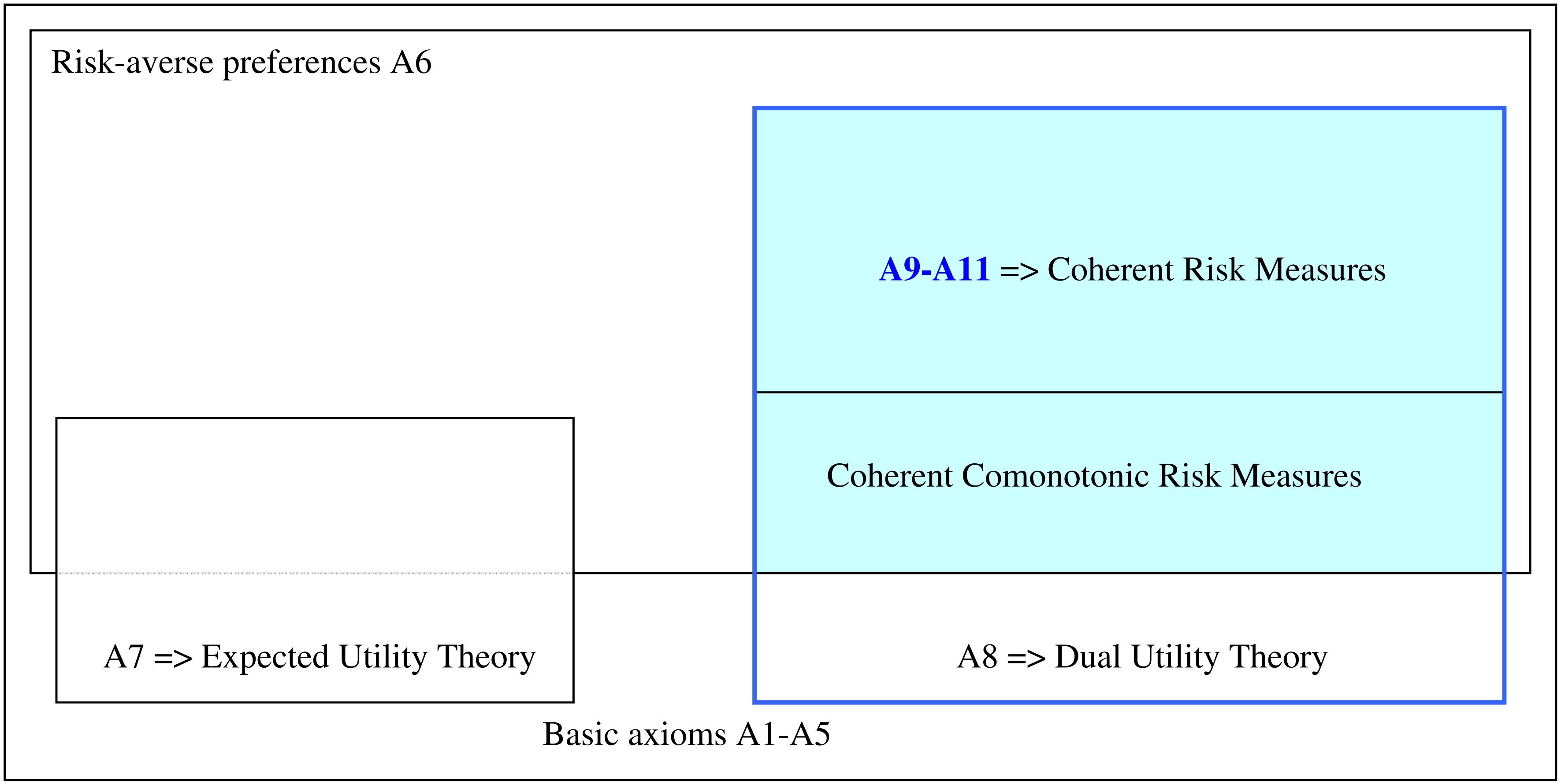
$\mathcal{R}(X)$ is coherent risk measure which is continuous and law-invariant

$X \succeq Y \Leftrightarrow \mathcal{R}(X) \leq \mathcal{R}(Y)$

$A8 \rightarrow A9$, $A10$, $A11$

comonotonic coherent risk measure $\rightarrow$ coherent risk measure.

Theories Interconnection



Paradoxes and dual paradoxes

Utility theory:

Possible: $X \prec 0$, $Y \prec 0$ but $1/2X + 1/2Y \succ 0$

Impossible: $X \succ 0$, $Y \succ 0$ but $1/2X + 1/2Y \prec 0$

Dual utility theory $\Leftrightarrow$ Coherent comonotonic risk measures

Possible: $X \succ 0$, $Y \succ 0$ but $1/2X \oplus 1/2Y \prec 0$

Impossible: $X \prec 0$, $Y \prec 0$ but $1/2X \oplus 1/2Y \succ 0$

Coherent risk measures

Possible: $X \succ 0$, $Y \succ 0$ but $1/2X \oplus 1/2Y \prec 0$

Possible: $X \prec 0$, $Y \prec 0$ but $1/2X \oplus 1/2Y \succ 0$

Bi-criteria Optimization

- Mean-Variance approach (Markowitz)
  Reward: EX
  Risk: $\sigma(X)$

- Mean-Risk models (Ruszczynski, Ogryczak)
  Reward: EX
  Risk: $\sigma_-(X)$, $\sigma_+(X)$, $MAD(X)$, $\ldots$

- Mean-Deviation analysis (Rockafellar et.al.)
  Reward: EX
  Risk: $\mathcal{D}(X)$

General Deviation Measures

- (D1) – **insensitivity to constant shift**
  $\mathcal{D}(X + C) = \mathcal{D}(X)$ *for all* X *and constants* C
- (D2) – **positive homogeneity**
  $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ *for all* X *and all* $\lambda > 0$
- (D3) – **subadditivity**
  $\mathcal{D}(X + X') \leq \mathcal{D}(X) + \mathcal{D}(Y)$ *for all* X *and* Y
- (D4) – **nonnegativity**
  $\mathcal{D}(X) \geq 0$ *(equality for constant* X*)*
- (D5) – **lower range dominated**
  $\mathcal{D}(X) \leq EX - \inf X$ *for all* X

Examples of Deviation Measures

- **Standard Deviation**

$$\sigma(X) = \left(E[X - EX]^2)\right)^{\frac{1}{2}}$$

- **Standard Semideviations**

$$\sigma_+(X) = (E[\max\{X - EX, 0\}^2])^{1/2}$$

$$\sigma_-(X) = (E[\max\{EX - X, 0\}^2])^{1/2}$$

- **Deviation measure from range**

$$EX - \inf X$$

- **Conditional Value-at-Risk (CVaR) deviation**

for $\alpha \in [0,1]$ $\quad \mathrm{CVaR}_\alpha^\Delta(X) = EX - \frac{1}{\alpha}\int_0^\alpha (-\mathrm{VaR}_p(X))dp$

Mean-Deviation Analysis

Mean-Deviation Analysis

$$V(EX, \mathcal{D}(X))$$

$$\mathcal{D}(X) = \sigma(X)$$

Mean-Variance Analysis

$$V(EX, \sigma(X))$$

$$V(m, d) = -m + d$$

Risk Measures

$$\mathcal{R}(X) = -EX + \mathcal{D}(X)$$

$\mathcal{D}$ **is lower range dominated**

$\Rightarrow \mathcal{R}$ **is coherent risk measure**

Mean-Deviation Analysis

Aim:

Axioms $A1 - A5$ + new axioms $\Rightarrow U(X) = V(EX, \mathcal{D}(X))$

$X \succeq Y \Leftrightarrow V(EX, \mathcal{D}(X)) \geq V(EY, \mathcal{D}(Y))$

Questions

- What conditions on $\mathcal{D}(X)$?
- What conditions on $V(m, d)$?
- What new axioms?

Axioms analysis

A11 Constant relative risk aversion

$X \succeq Y, \lambda > 0 \Rightarrow \lambda X \succeq \lambda Y$

(0 or 100) $\succ$ (40 for sure)

$\Downarrow$?

(0 or 10,000) $\succ$ (4,000 for sure)

A10 Constant absolute risk aversion

$X \succeq Y, C$ *constant* $\Rightarrow X + C \succeq Y + C$

Y with $EY = 10$ $\sim$ 1 for sure

$\Downarrow$?

$Y + 1$ $\sim$ 2 for sure

Axioms relaxation

Idea: Axioms for coherent risk measures $+$ condition (EX=EY) $=$ new set of axioms (relaxed).

A12 **Convexity with $EX = EY$**
X *and* Y *comonotonic,* $X \sim Y$ *and* $EX = EY \Rightarrow$
$\lambda X + (1-\lambda)Y \succeq Y$

A13 **Constant absolute risk aversion with $EX = EY$**
$X \succeq Y$ *and* $EX = EY$, C *constant* $\Rightarrow X + C \succeq Y + C$

A14 **Constant relative risk aversion with $EX = EY$**
$X \succeq Y$ *and* $EX = EY$, $\lambda > 0 \Rightarrow \lambda X \succeq \lambda Y$

- A9 + (EX=EY) $\rightarrow$ A12
- A10 + (EX=EY) $\rightarrow$ A13
- A11 + (EX=EY) $\rightarrow$ A14

Conditions on Deviation Measure

Let $\mathcal{D}(X)$ be

- Continuous (natural)
- Law-invariant (natural)
- Weakly lower-range dominated (additional)

Definition: $\mathcal{D}(X)$ is weakly lower-range dominated $\Leftrightarrow$

$$\sup_{X \neq C} \frac{\mathcal{D}(X)}{EX - \inf X} = K < \infty \tag{1}$$

$K \leq 1 \Leftrightarrow \mathcal{D}(X)$ is lower-range dominated

Conditions on V(m,d)

$$X \succeq Y \Leftrightarrow V(EX, \mathcal{D}(X)) \geq V(EY, \mathcal{D}(Y))$$

Let $V(m, d)$ be continuous and satisfies

$V1$ Strictly increasing in m (natural)

$V2$ Strictly decreasing in d (natural)

$V3$ Normalization: $V(m, 0) = m$ (technical)

$V4$ $\exists K > 0:$ such that $\forall$ m, $d \geq 0$ $a \geq 0$ we have

$V(m, d) \leq V(m + a, d + aK)$. (additional)

Main result

Theorem. The following are equivalent.

- $\succeq$ satisfies axioms $A1 - A6$ and $A12 - A14$.
- $\exists\ K > 0$, $\mathcal{D}(X)$ and $V(m,d)$ such that

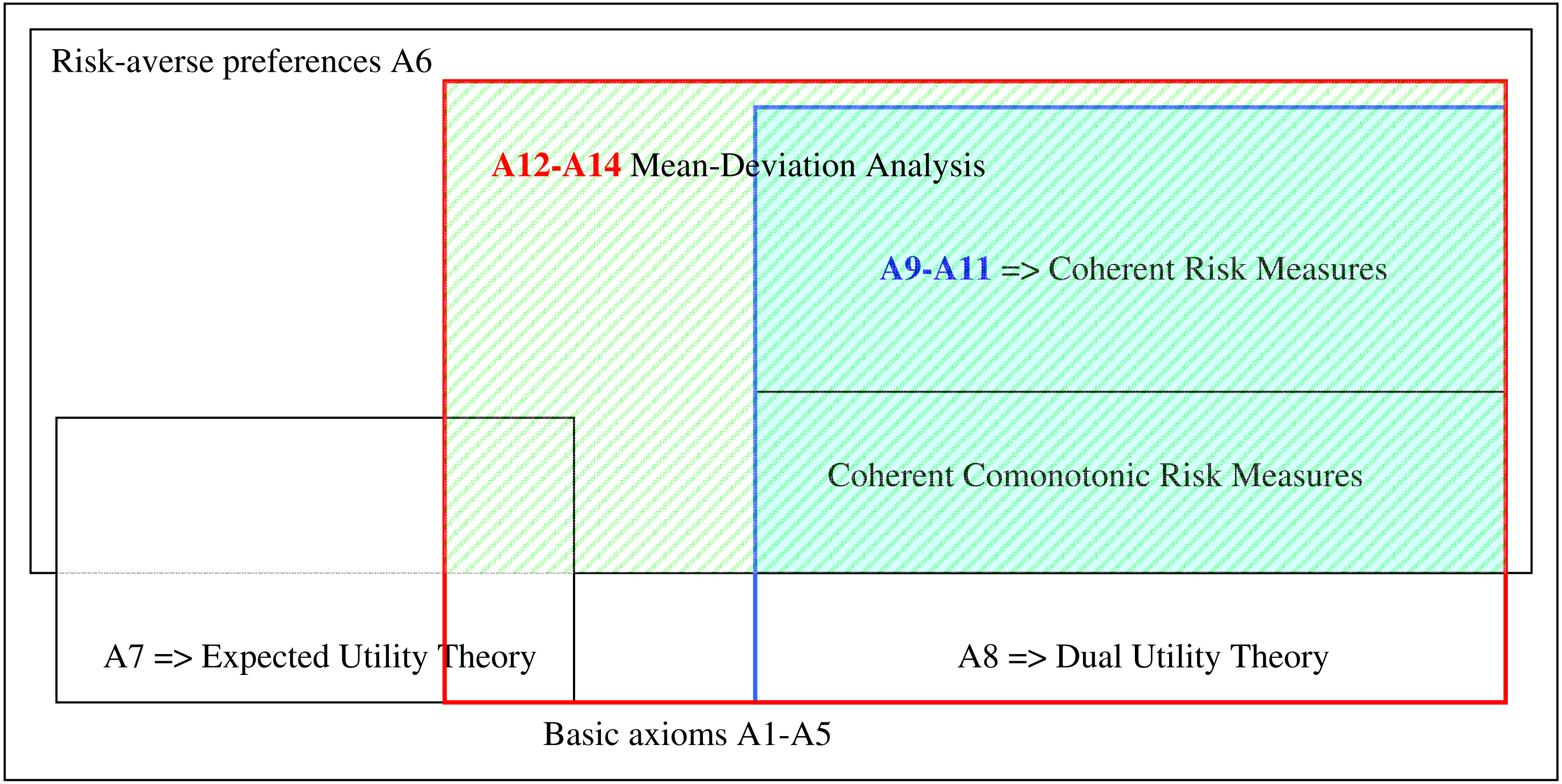
  $X \succeq Y \Leftrightarrow V(EX, \mathcal{D}(X)) \geq V(EY, \mathcal{D}(Y))$

where $\mathcal{D}(X)$ is a deviation measure which is

- continuous
- law-invariant
- weakly lower-range dominated with constant K

and $V(m,d)$ satisfies $V1 - V3$ and $V4$ with constant K

Theories Interconnection



Concluding Remarks

- **Axioms A1-A5 $\Rightarrow$ Numerical representation of preferences**
  $U(X)$
- **Axioms A1-A5 + A7 $\Rightarrow$ Expected Utility Theory**
  $U(X) = Eu(X)$
- **Axioms A1-A5 + A8 $\Rightarrow$ Dual Utility Theory**
- **Axioms A1-A6 + A8 $\Rightarrow$ Coherent Comonotonic Risk Measures**
  $U(X) = -\mathcal{R}(X)$
- **Axioms A1-A6 + A9-A11 $\Rightarrow$ Coherent Risk Measures**
  $U(X) = -\mathcal{R}(X)$
- **Axioms A1-A6 + A12-A14 $\Rightarrow$ Mean-Deviation Analysis**
  $U(X) = V(EX, \mathcal{D}(X))$