Optimization under Stochastic Ordering Constraints

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Outline

- Stochastic Orders, Stochastic Dominance
- Dominance-Constrained Optimization
- Optimality Conditions and Duality
- Extensions and Generalizations
- Applications

Stochastic Dominance

Introduced in statistics in 1947 by Mann and Whitney
Blackwell (1953), Lehmann (1955)

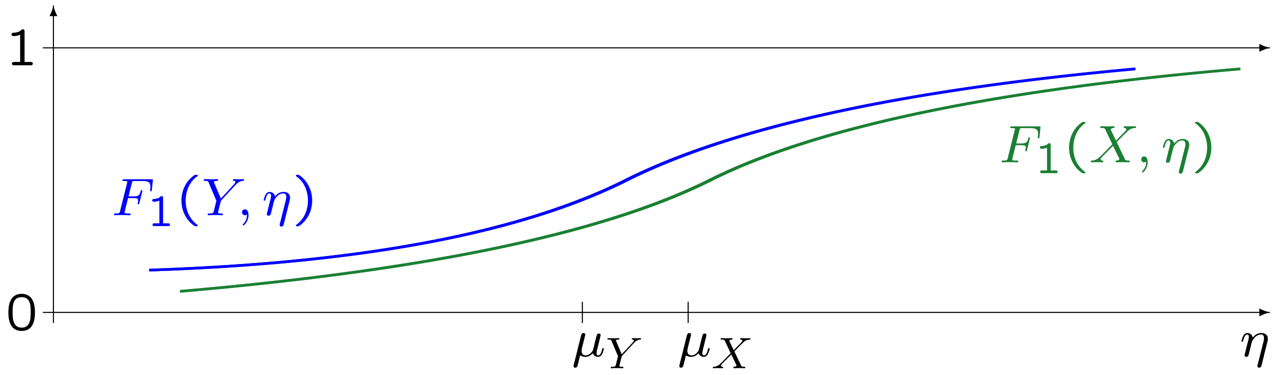
Distribution Functions

$$\begin{aligned} F_1(X;\eta) &= \int_{-\infty}^{\eta} P_X(dt) = \mathbb{P}\{X \leq \eta\} \quad \text{for all } \eta \in \mathbb{R} \\ F_k(X;\eta) &= \int_{-\infty}^{\eta} F_{k-1}(X;t)\,dt \quad \text{for all } \eta \in \mathbb{R}, \qquad k = 2, 3, \ldots \end{aligned}$$

kth order Stochastic Dominance

$$X \succeq_{(k)} Y \quad \Leftrightarrow \quad F_k(X,\eta) \leq F_k(Y,\eta) \quad \text{for all } \eta \in \mathbb{R}$$

First Order Stochastic Dominance: Usual Stochastic Order



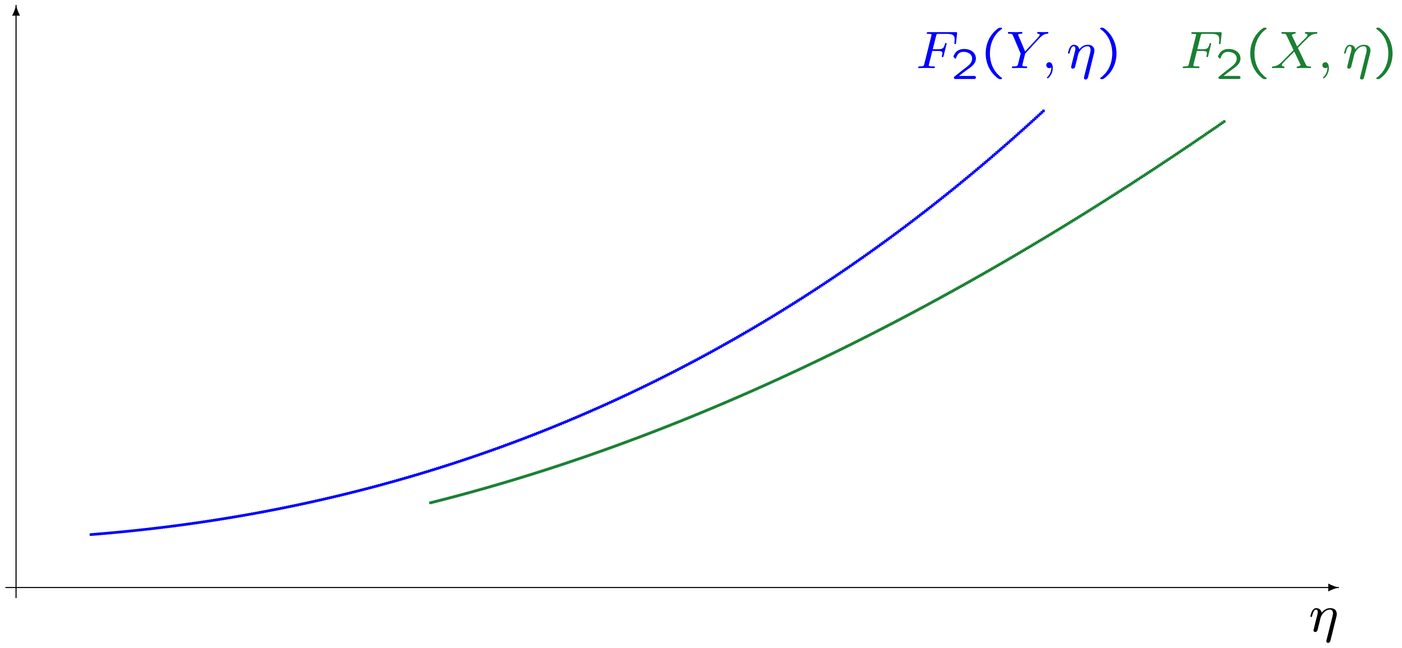
First order dominance $\equiv$ Continuum of probability inequalities

$$X \succeq_{(1)} Y \quad \Leftrightarrow \quad F_{(-1)}(X;p) \geq F_{(-1)}(Y;p) \quad \text{for all} \quad 0 < p < 1$$

$$F_{(-1)}(X;p) = \inf\{\eta : F_1(X;\eta) \geq p\}$$

Second-Order Stochastic Dominance

$$F_2(X,\eta) = \int_{-\infty}^{\eta} F_1(X,t)\,dt = \mathbb{E}(\eta - X)_+ \text{ for all } \eta \in \mathbb{R}$$



Relation to Utility Functions in Economics

Expected Utility Models (von Neumann and Morgenstern, 1947)

$$\max_{z\in Z}\ \mathbb{E}\left[u(G(z))\right]$$

Z - subset of an appropriate vector space $\mathcal{Z}$

$(\Omega, \mathcal{F}, P)$ - probability space

$G : \mathcal{Z} \to \mathcal{L}_1(\Omega, \mathcal{F}, P; \mathbb{R})$ - random outcome

$u(\cdot)$ - utility function (non-decreasing)

Risk-Averse Consistency:

Quirk and Saposnik (1962)

$$X \succeq_{(1)} Y \Leftrightarrow \mathbb{E}\,u(X) \geq \mathbb{E}\,u(Y) \quad \forall \textit{ nondecreasing } u(\cdot)$$

Hadar and Russell (1969)

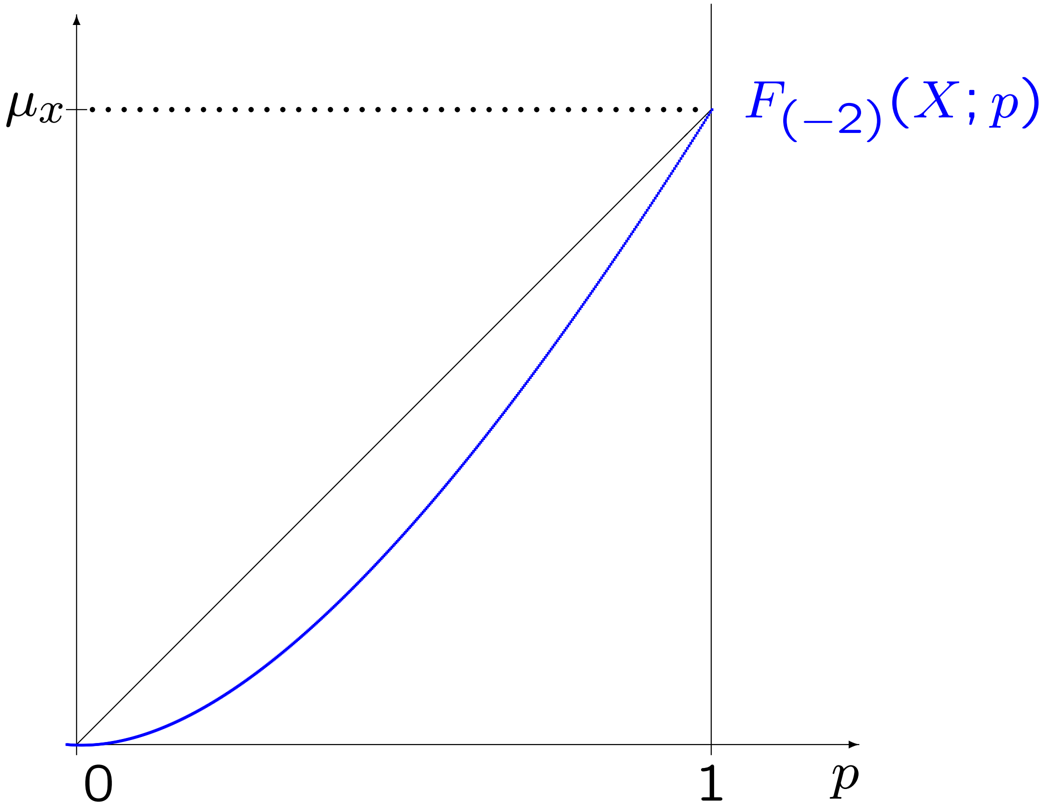
$$X \succeq_{(2)} Y \Leftrightarrow \mathbb{E}\,u(X) \geq \mathbb{E}\,u(Y) \quad \forall \text{ nondecreasing concave } u(\cdot)$$

Lorenz Order (Max Otto Lorenz, 1905)

Absolute Lorenz function $F_{(-2)}(X;\cdot) : \mathbb{R} \to \bar{\mathbb{R}}$:

$$F_{(-2)}(X;p) = \int_0^p F_{(-1)}(X;t)\,dt \quad \text{for all} \quad 0 < p \le 1,$$

$F_{(-2)}(X;0) = 0$ and $F_{(-2)}(X;p) = +\infty$ for $p \notin [0,1]$



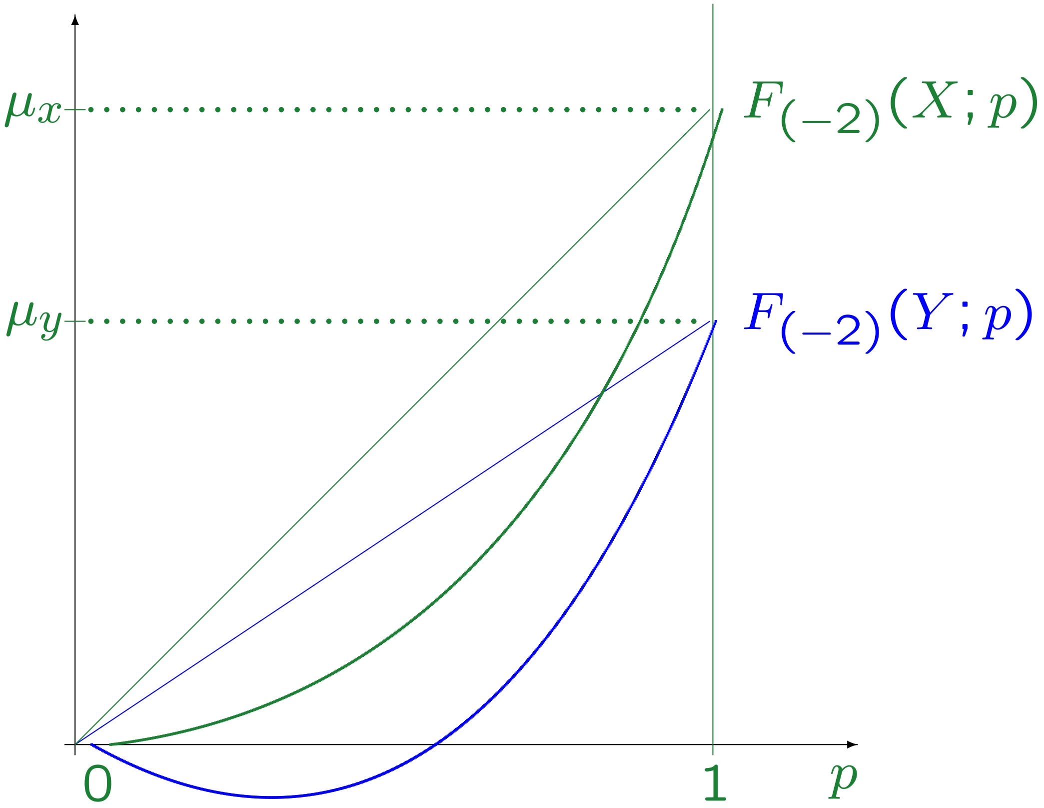
$$X \succeq_L Y \Leftrightarrow F_{(-2)}(X;p) \ge F_{(-2)}(Y;p) \quad \text{for all} \quad 0 \le p \le 1$$

Characterization of stochastic dominance by Lorenz functions

Fenchel conjugate function of F: $F^*(p) = \sup_u \{pu - F(u)\}$

$$F_{(-2)}(X;\cdot) = [F_2(X;\cdot)]^* \quad \text{and} \quad F_2(X;\cdot) = [F_{(-2)}(X;\cdot)]^*$$

Ogryczak - Ruszczynski (2002)



$$X \succeq_{(2)} Y \quad \Leftrightarrow \quad F_{(-2)}(X;p) \geq F_{(-2)}(Y;p) \quad \text{for all} \quad 0 \leq p \leq 1.$$

Relation to rank dependent utility functions

Rank Dependent Utility (Quiggin 1982, Schmeidler 1986–89, Yaari 1987)

The set $\mathcal{W}_1$ contains all continuous nondecreasing functions $w : [0,1] \to \mathbb{R}$.

The set $\mathcal{W}_2 \subset \mathcal{W}_1$ contains all concave subdifferentiable at 0 functions.

Theorem 1 [DD, A. Ruszczyński, 2005]
(i) For any two random variables $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ the relation $X \succeq_{(1)} Y$ holds true if and only if for all $w \in \mathcal{W}_1$

$$\int_0^1 F_{(-1)}(X;p)\, dw(p) \geq \int_0^1 F_{(-1)}(Y;p)\, dw(p) \tag{1}$$

(ii) $X \succeq_{(2)} Y$ holds true if and only if (1) is satisfied for all $w \in \mathcal{W}_2$.

Stochastic Orders Generalizations

Integral Stochastic Orders

$$X \succeq_{\mathcal{F}} Y \;\Leftrightarrow\; \mathbb{E}\,u(X) \geq \mathbb{E}\,u(Y) \quad \forall\, u(\cdot) \in \mathcal{F}$$

Collection of functions $\mathcal{F}$ is the generator of the order.

Multivariate Orders, Orders of Sequences

$\hookrightarrow$ talk by Andrzej Ruszczyński tomorrow, March 31.

Dominance Relation in Optimization

$$\begin{aligned} &\max\ \mathbb{E}\Big[H(z)\Big] \\ &\text{subject to } G_i(z) \succeq_{(k_i)} Y_i, \quad i = 1..m \\ &z \in Z \end{aligned}$$

Y_i - benchmark random outcome

The dominance constraints reflect risk aversion:
$G_i(z)$ is preferred over Y_i by all risk-averse decision makers

Introduced by DD, A.Ruszczyński, 2003

Portfolio Example

Assets $j = 1, \ldots, n$ with random return rates R_j

Decision variables z_j, $j = 1, \ldots, n$, Z -simplex

Portfolio return rate $H(z) = G(z) = \sum_{j=1}^{n} z_j R_j$

Reference return rate Y (e.g. index, existing portfolio, etc.)

$$\text{max } \mathbb{E}\Big[\sum_{j=1}^{n} z_j R_j \Big]$$

$$\text{subject to } \sum_{j=1}^{n} z_j R_j \succeq_{(2)} Y$$

$$z \in Z$$

All statements are equivalent:

$$\sum_{j=1}^{n} z_j R_j \succeq_{(2)} Y$$

$$F_{(-2)}\Big(\sum_{j=1}^{n} z_j R_j ; p\Big) \geq F_{(-2)}(Y;p) \quad \text{for all} \quad p \in [0,1]$$

continuum of CVaR (integrated chance) constraints for risk levels $p \in [0,1]$

$$\mathbb{E}u\Big(\sum_{j=1}^{n} z_j R_j\Big) \geq \mathbb{E}u(Y)$$

for <u>all</u> concave nondecreasing functions u (von Neuman-Morgenstern utility)

$$\int_0^1 F_{(-1)}\Big(\sum_{j=1}^{n} z_j R_j ; p\Big)\, dw(p) \geq \int_0^1 F_{(-1)}(Y;p)\, dw(p)$$

for <u>all</u> concave nondecreasing functions w (rank dependent utility)

Sets defined by dominance relation

For all $k \geq 1$
Y - benchmark outcome in $\mathcal{L}_k(\Omega, \mathcal{F}, P)$
$[a, b] \subseteq \mathbb{R}$
we define the set

$$A_k(Y; [a, b]) = \{X \in \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P) : X \succeq_{(k)} Y \text{ in } [a, b]\}$$

Theorem 2 The set $A_k(Y; [a, b])$ is convex and closed for all $[a, b]$, all Y, and $k \geq 2$. Its recession cone has the form

$$A_k^{\infty}(Y; [a, b]) = \{H \in \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P) : H \geq 0 \text{ a.s. on } [a, b]\}$$

The set $A_1(Y; [a, b])$ is closed and

$$A_k(Y; [a, b]) \subseteq A_{k+1}(Y; [a, b]) \quad \text{for all } k \geq 1.$$

$A_k(Y; [a, b])$ is a cone pointed at Y if and only if Y is a constant in $[a, b]$.

Theorem 3 Assume that $\Omega = \{1, \dots, N\}$, $\mathcal{F}$ is the set of all subsets of Ω and $\mathbb{P}[k] = 1/N$, $k = 1, \dots, N$. If Y is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ then

$$A_2(Y; \mathbb{R}) = \operatorname{conv} A_1(Y; \mathbb{R})$$

Theorem 4 Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ admits a continuous distribution. Then

$$A_2(Y; \mathbb{R}) = \overline{\operatorname{conv}}\, A_1(Y; \mathbb{R})$$

The result is not true for general probability spaces

Inverse dominance constrained set

$$B(Y; [\alpha, \beta]) = \left\{ X \in \mathcal{L}^1(\Omega, \mathcal{F}, P) : F_{(-2)}(X; p) \geq F_{(-2)}(Y; p) \;\; \forall\, p \in [\alpha, \beta] \right\}$$

Theorem 5

- The set $B(Y; [\alpha, \beta])$ is convex and closed.
- If $[a, b]$ contains all p-quantiles of Y for $p \in [\alpha, \beta]$, then

$$A_2(Y; [a, b]) \subset B(Y; [\alpha, \beta]).$$

Abstract Problem Formulation

$$\begin{aligned} &\text{max } f(X) \\ &\text{subject to } F_2(X;\eta) \le F_2(Y;\eta) \quad \text{for all } \eta \in [a,b] \qquad \textbf{(P)} \\ &X \in C \end{aligned}$$

C - convex set in $\mathcal{L}_1(\Omega, \mathcal{F}, P)$

f - concave functional

Y - reference outcome in $\mathcal{L}_1(\Omega, \mathcal{F}, P)$

Key Results

- Concave nondecreasing functions $u(\cdot)$ play the role of Lagrange multipliers associated with the dominance constraint

- The function

$$L(X,u) = f(X) + \mathbb{E}\Big[u(X)\Big] - \mathbb{E}\Big[u(Y)\Big]$$

plays the role of the Lagrangian

The set $\mathcal{U}_2$ of multipliers $u(\cdot)$ is defined as follows:

$u(\cdot)$ is concave and nondecreasing

$u(t) = 0$ for all $t \geq b$

$u(t) = u(a) + c(t - a)$, with some $c > 0$, for all $t \leq a$

Definition 1

Problem satisfies the Uniform Dominance Condition if there exists a point $\tilde{X} \in C$ such that

$$\inf_{\eta \in [a,b]} \left\{ F_2(Y;\eta) - F_2(\tilde{X};\eta) \right\} > 0.$$

Optimality Conditions

Theorem 6

1. Assume that the uniform dominance condition is satisfied. If $\hat{X}$ is an optimal solution of problem **(P)** then there exists a function $\hat{u} \in \mathcal{U}_2$ such that

$$L(\hat{X},\hat{u}) = \max_{X \in C} L(X,\hat{u}) \tag{2}$$

$$\mathbb{E}\big[\hat{u}(\hat{X})\big] = \mathbb{E}\big[\hat{u}(Y)\big] \tag{3}$$

2. If for a function $\hat{u} \in \mathcal{U}_2$ an optimal solution $\hat{X}$ of (2) satisfies (3) and the dominance constraint, then $\hat{X}$ is an optimal solution of problem **(P)**.

Key points of the proof: Define $\Gamma : \mathcal{L}_1(\Omega,\mathcal{F},P) \to \mathcal{C}([a,b])$ as

$$\Gamma(X)(\eta) = F_2(Y;\eta) - F_2(X;\eta) \quad \text{for} \quad \eta \in [a,b]$$

Correspondence between nonnegative measures μ from $\mathbf{rca}([a,b])$ and functions $u \in \mathcal{U}_2$ such that

$$\int_a^b F_2(X;\eta)\mu(d\eta) = -\mathbb{E}\big[u(X)\big].$$

Duality

The Lagrangian

$$L(X,u) = f(X) + \mathbb{E}[u(X)] - \mathbb{E}[u(Y)]$$

The dual functional

$$D(u) = \sup_{X\in C} L(X,u)$$

The dual problem

$$\min_{u\in\mathcal{U}_2} D(u)$$

Theorem 7

Assume that the uniform dominance condition is satisfied and problem **(P)** has an optimal solution. Then the dual problem has an optimal solution and the optimal values of both problems coincide. Furthermore, the set of optimal solutions of the dual problem is the set of functions $\hat{u} \in \mathcal{U}_2$ satisfying (2)–(3) for an optimal solution $\hat{X}$ of problem **(P)**.

Generalization: Higher Order Dominance

$$\begin{aligned} &\text{max } f(X) \\ &\text{subject to } X \succeq_{(k)} Y \text{ in } [a,b] \qquad\qquad (\mathbf{Pk}) \\ &\qquad X \in C \subseteq \mathcal{L}_{k-1}(\Omega,\mathcal{F},P) \end{aligned}$$

The set $\mathcal{U}_k$ contains all functions $u : \mathbb{R} \to \mathbb{R}$, for which there exists a non-negative, nonincreasing, left-continuous and bounded function
$\varphi : [a,b] \to \mathbb{R}$ such that

$$\begin{aligned} &u^{(k-1)}(t) = (-1)^k \varphi(t), \quad \text{for a.a. } t \in [a,b] \\ &u^{(k-1)}(t) = (-1)^k \varphi(a), \quad \text{for } t < a \\ &u(t) = 0, \quad \text{for } t \geq b \\ &u^{(i)}(b) = 0, \quad i = 1, \ldots, k-2 \text{ if } k \geq 2. \end{aligned}$$

The Uniform k-Dominance Condition: there exists $\tilde{X} \in C$ such that

$$\inf_{\eta \in [a,b]} \left\{ F_k(Y;\eta) - F_k(\tilde{X};\eta) \right\} > 0$$

Theorem 8

Assume that the Uniform k-Dominance Condition is satisfied. If $\hat{X}$ is an optimal solution of problem (Pk) then there exists a function $\hat{u} \in \mathcal{U}_k$ such that

$$L(\hat{X},\hat{u}) = \max_{X \in C} L(X,\hat{u}) \tag{4}$$

$$\mathbb{E}\left[\hat{u}(\hat{X})\right] = \mathbb{E}\left[\hat{u}(Y)\right] \tag{5}$$

Conversely, if for some function $\hat{u} \in \mathcal{U}_k$ an optimal solution $\hat{X}$ of (4) satisfies the dominance constraint in (Pk) and (5), then $\hat{X}$ is an optimal solution of problem (Pk).

Generalization: Nonlinear Second Order Dominance Constraints

Problem (SSD)

$$\begin{array}{rllll} \text{max} & \mathbb{E}[H(z)] & & & \\ \text{subject to} & G_i(z) \succeq_{(2)} Y_i & \text{in} & [a_i, b_i] & i = 1, \ldots, m \\ & z \in Z & & & \end{array}$$

Z - convex subset of a separable locally convex Hausdorff vector space $\mathcal{Z}$

G_i and H – continuous operators from $\mathcal{Z}$ to the space $\mathcal{L}_1(\Omega, \mathcal{F}, P; \mathbb{R})$

G_i and H are concave in the following sense: for P-almost all $\omega \in \Omega$ the functions $[G_i(\cdot)](\omega)$, $i = 1, \ldots, m$, and $[H(\cdot)](\omega)$ are concave and continuous

We denote the convex cone

$$\mathcal{U}_2^m = \mathcal{U}_2([a_1, b_1]) \times \cdots \times \mathcal{U}_2([a_m, b_m])$$

The Lagrangian, $L : \mathcal{Z} \times \mathcal{U}_2^m \to \mathbb{R}$,
associated with problem (SSD) takes on the form:

$$L(z, u) := \mathbb{E}\Big[H(z) + \sum_{i=1}^{m} \Big(u_i(G_i(z)) - u_i(Y_i)\Big)\Big]$$

Optimality Conditions

Definition 2 Problem (SSD) satisfies the Uniform Dominance Condition if there exists a point $\tilde{z} \in Z$ such that

$$\inf_{\eta \in [a_i, b_i]} \Big\{ F_2(Y_i; \eta) - F_2(G_i(\tilde{z}); \eta) \Big\} > 0, \quad i = 1, \dots, m$$

Theorem 9

Assume that the uniform dominance condition is satisfied. If $\hat{z}$ is an optimal solution of problem (SSD) then there exist $\hat{u} \in \mathcal{U}_2^m$ such that

$$L(\hat{z}, \hat{u}) = \max_{z \in Z} L(z, \hat{u}) \tag{6}$$

$$\mathbb{E}\Big[\hat{u}_i(G_i(\hat{z})\Big] = \mathbb{E}\Big[\hat{u}_i(Y_i)\Big], \quad i = 1, \dots, m \tag{7}$$

Conversely, if for some function $\hat{u} \in \mathcal{U}_2^m$ an optimal solution $\hat{z}$ of (6) is feasible for (SSD) and satisfies (7), then $\hat{z}$ is an optimal solution of (SSD).

Problem with Inverse Stochastic Dominance Constraint

$$\begin{aligned} &\max \ f(X) \\ &\text{subject to } F_{(-2)}(X;p) \geq F_{(-2)}(Y;p) \quad \forall\, p \in [\alpha,\beta] \qquad\qquad \textbf{PID} \\ &\phantom{\text{subject to }} X \in C \subseteq \mathcal{L}_1(\Omega,\mathcal{F},P) \end{aligned}$$

$$[\alpha,\beta] \subset (0,1)$$

Main Results

- Concave nondecreasing functions $w : [0,1] \to \mathbb{R}$ play the role of Lagrange multipliers associated with the inverse dominance constraint
- The function $\Phi : C \times \mathcal{W}_2([\alpha,\beta]) \to \mathbb{R}$

$$\Phi(X,w) = f(X) + \int_0^1 F_{(-1)}(X;p)\, dw(p) - \int_0^1 F_{(-1)}(Y;p)\, dw(p)$$

plays the role of the Lagrangian

The set $\mathcal{W}_2([\alpha,\beta])$ contains functions $w : [0,1] \to \mathbb{R}$ such that:

$w(\cdot)$ is concave and nondecreasing;
$w(p) = 0$ for all $p \in [\beta,1]$;
$w(p) = w(\alpha) + c(p-\alpha)$, with some $c > 0$, for all $p \in [0,\alpha]$.

Problem (PID) satisfies the *uniform inverse dominance condition* if there exists a point $\tilde{X} \in C$ such that

$$\inf_{p \in [\alpha,\beta]} \left\{ F_{(-2)}(\tilde{X};p) - F_{(-2)}(Y;p) \right\} > 0$$

Theorem 10 Assume the uniform inverse dominance condition.
If $\hat{X}$ is an optimal solution of problem (PID), then there exists a function $\hat{w} \in \mathcal{W}_2([\alpha,\beta])$ such that

$$\Phi(\hat{X},\hat{w}) = \max_{X \in C} \Phi(X,\hat{w}) \tag{8}$$

and

$$\int_0^1 F_{(-1)}(\hat{X};p)\, d\hat{w}(p) = \int_0^1 F_{(-1)}(Y;p)\, d\hat{w}(p) \tag{9}$$

If for some function $\hat{w} \in \mathcal{W}_2([\alpha,\beta])$ an optimal solution $\hat{X}$ of (8) satisfies the inverse dominance constraints and (9), then $\hat{X}$ is an optimal solution of problem (PID).

First Order Dominance-Constrained Problem

$$\begin{aligned} &\max\ \mathbb{E}\Big[H(z)\Big] \\ \textbf{(FSD)} \qquad &\text{subject to } \mathbb{P}\Big[G(z) \le \eta\Big] \le \mathbb{P}\Big[Y \le \eta\Big], \quad \eta \in [a,b] \\ &z \in Z \end{aligned}$$

Challenge: Constraints on probability introduce non-convexity

Necessary conditions of optimality in differential form can be obtained under additional differentiability assumptions for H and G and differential form of the constraint qualification.

The convex cone $\mathcal{U}_1([a_i, b_i])$ contains of functions $u(\cdot)$ such that:

$$\begin{aligned} &u(\cdot) \text{ is nondecreasing and left continuous;} \\ &u(t) = 0 \text{ for all } t \ge b_i; \\ &u(t) = u(a), \text{ for all } t \le a_i. \end{aligned}$$

If $\mathcal{Z} = \mathbb{R}^n$, then the optimal utility function $\hat{u}(\cdot) \in \mathcal{U}_1$ is piecewise constant with at least one and at most $n+2$ jump points.

Sufficient conditions of optimality can be obtained under additional generalized convexity assumptions.

Discrete Distributions

$\Omega = \{\omega_1, \ldots, \omega_n\}$ with probabilities $p_j = P(\{\omega_j\})$, $j = 1, \ldots, n$
$J = \{1, \ldots, n\}$, $I = \{1, \ldots, m\}$

$$h_j(z) = H(z)(\omega_j),\ \ g_{ij}(z) = G_i(z)(\omega_j),\ \ y_{ij} = Y_i(\omega_j),\ \ x_{ij} = X_i(\omega_j)$$

Problem with Nonlinear Second Order Dominance constraint

$$\begin{aligned}
\text{max}\ \ & \sum_{j=1}^{n} p_j h_j(z) \\
\text{subject to}\ \ & \sum_{j=1}^{n} p_j (y_{ik} - x_{ij})_+ \le \sum_{j=1}^{n} p_j (y_{ik} - y_{ij})_+, \quad i \in I, \quad k \in J \\
& x_{ik} \le g_{ik}(z), \quad i \in I, \quad k \in J \\
& z \in Z
\end{aligned}$$

Optimality and duality conditions remain valid under the Slater Condition.

The utility functions corresponding to the ith group of dominance constraints are concave nondecreasing functions $u_i(\cdot)$ which are piecewise–linear with break points at y_{ik}, $k \in J$.

Application to Portfolio Optimization

Assets $j = 1, \ldots, N$ with random returns R_j

Decision variables z_j, $j = 1, \ldots, N$, Z -simplex

Portfolio return $G(z) = \sum_{j=1}^{N} z_j R_j$

Reference return Y (e.g. index, existing portfolio, etc)

$$\begin{aligned} \text{max } & \mathbb{E}\Big[\sum_{j=1}^{n} z_j R_j\Big] \\ \text{subject to } & \sum_{j=1}^{n} z_j R_j \succeq_{(2)} Y, \quad i = 1..m \\ & z \in Z \end{aligned}$$

719 real-world assets, 616 possible realizations of their joint returns.

Benchmarck Portfolios:

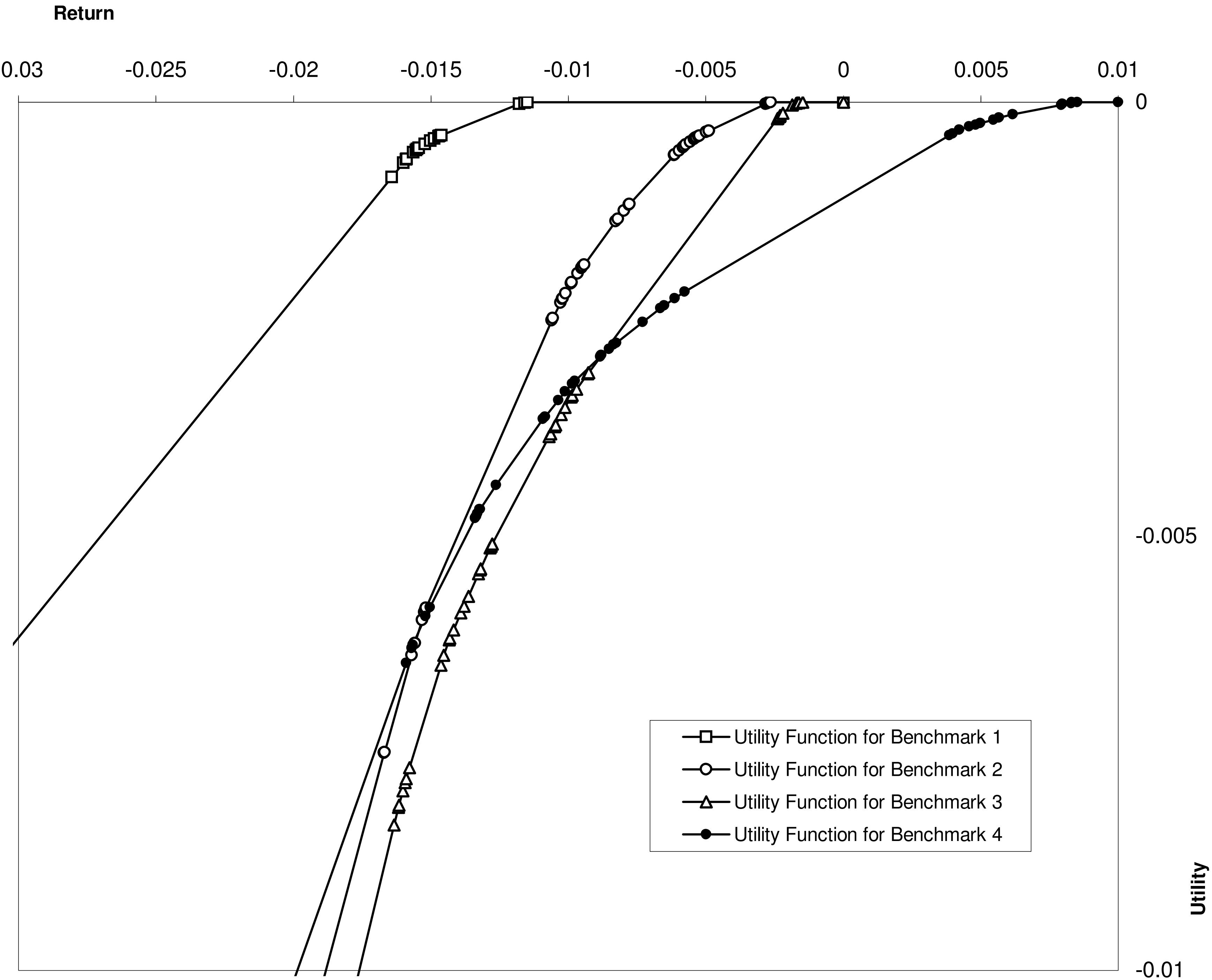
equally weighted indexes composed of N of our assets

Portfolio 1 corresponds to $N = 26$,

Portfolio 2 corresponds to $N = 54$,

Portfolio 3 corresponds to $N = 82$,

and Portfolio 4 corresponds to $N = 200$.


Return
-0.03
-0.025
-0.02
-0.015
-0.01
-0.005
0
0.005
0.01
0
-0.005
-0.01
Utility
Utility Function for Benchmark 1
Utility Function for Benchmark 2
Utility Function for Benchmark 3
Utility Function for Benchmark 4

Further Applications

- Electricity markets: D. Berleant, M. Dancre, G. Sheble, 2005
- Inverse models and forecasting: Compare the forecast errors via stochastic dominance and design data collection for model calibration
- Medicine: radiation therapy designs

First Order Dominance-Constrained Problem

$$
\textbf{(FSD)}\qquad
\begin{aligned}
&\max\ \mathbb{E}\Big[H(z)\Big]\\
&\text{subject to } \mathbb{P}\Big[G_i(z)\le\eta\Big]\le\mathbb{P}\Big[Y_i\le\eta\Big],\quad \eta\in[a_i,b_i],\quad i=1,\dots,m\\
&z\in Z
\end{aligned}
$$

(i) Z is a convex closed subset of a separable Banach space $\mathcal{Z}$

(ii) $H\colon \mathcal{Z}\to\mathcal{L}_1(\Omega,\mathcal{F},\mathbb{P};\mathbb{R})$ is continuously differentiable on Z in Fréchet sense

(iii) $G_i\colon \mathcal{Z}\to\mathcal{L}_0(\Omega,\mathcal{F},\mathbb{P};\mathbb{R})$ are continuous operators.

(iv) For every $z\in Z$ the distribution of $G_i(z)$ has a density $\varphi_i(\cdot,z)$, which is continuously differentiable in z for every x, and $D_z\varphi_i(\cdot,z)$ is locally bounded at all $z\in Z$ by an integrable function $K_i(\cdot;z)$

(v) $Y_i\in\mathcal{L}_1$ are benchmark outcomes with continuous distribution functions

Challenge: Constraints on probability introduce non-convexity

Necessary Conditions of Optimality

The normal cone to the set $Z \subseteq \mathcal{Z}$

$$N_Z(z) = \{d \in \mathcal{Z}^* : \langle d, y - z\rangle \le 0 \text{ for all } y \in Z\}$$

Differentiation of the constraint (probability) function:

$$S_i(z;\eta) := D_z\Big[\mathbb{P}[G_i(z) \le \eta]\Big] = \int_{-\infty}^{\eta} D_z\varphi_i(x,z)\ dx$$

The convex cone $\mathcal{U}_1([a_i,b_i])$ contains of functions $u(\cdot)$ such that:

$u(\cdot)$ is nondecreasing and left continuous;
$u(t) = 0$ for all $t \ge b_i$;
$u(t) = u(a)$, for all $t \le a_i$.

The Lagrangian $L : \mathcal{Z} \times \mathcal{U}_1^m \to \mathbb{R}$ with $\mathcal{U}_1^m = \mathcal{U}_1([a_1,b_1]) \times \cdots \times \mathcal{U}_1([a_m,b_m])$

$$L(z,u) := \mathbb{E}\left[H(z) + \sum_{i=1}^{m}\Big(u_i(G_i(z)) - u_i(Y_i)\Big)\right]$$

Differential uniform dominance condition (DUDC) at the point $\hat{z} \in Z$: there exists $z^0 \in Z$ such that

$$\max_{a_i \le \eta \le b_i} \Big\{ \mathbb{P}\big[G_i(\hat{z}) \le \eta\big] + S_i(\hat{z};\eta)(z^0 - \hat{z}) - \mathbb{P}\big[Y_i \le \eta\big] \Big\} < 0, \quad i = 1, \ldots, m$$

Theorem 4: Assume that $\hat{z}$ is an optimal solution of problem (FSD) and that the DUDC is satisfied at $\hat{z}$. Then there exists a function $\hat{u} \in \mathcal{U}_1^m$, $\hat{u} \neq 0$, such that

$$D_z L(\hat{z}, \hat{u}) \in N_Z(\hat{z}) \tag{10}$$

$$\mathbb{E}\big[\hat{u}_i(G_i(z))\big] = \mathbb{E}\big[\hat{u}_i(Y_i)\big], \quad i = 1..m \tag{11}$$

The set $\hat{U}$ of functions in $\mathcal{U}$ satisfying (10)–(11) for the local minimum $\hat{z}$ is convex, bounded and weakly* closed in the following sense:
if $u^k \in \hat{U}$ and $u \in \mathcal{U}$ are such that

$$\lim_{k \to \infty} \mathbb{E}\big[u_i^k(X)\big] = \mathbb{E}\big[u_i(X)\big] \quad \text{for all real random variables } X \text{ and all } i,$$

then $u \in \hat{U}$.

Critical targets in finite dimensions

Suppose that $\mathcal{Z} = \mathbb{R}^n$.

Theorem 5: Assume that $\hat{z}$ is an optimal solution and that the DUDC is satisfied at $\hat{z}$. Then there exist piecewise constant functions $w_i(\cdot) \in \mathcal{U}_i$ satisfying the necessary optimality conditions (10)–(11) such that $w_i(\cdot)$, $i = 1..m$, have in total at least one and at most $n + 2$ jump points.

Conclusion: There exist at most $n + 2$ target values η_{ik} and target probabilities $v_{ik} = \mathbb{P}\Big[Y_i \le \eta_{ik}\Big]$, $k = 1..k_i$, $i = 1..m$ and $1 \le \sum_{i=1}^{m} k_i \le n + 2$, which are critical for Problem (C).

$$\begin{aligned} &\max\ \mathbb{E}\Big[H(z)\Big] \\ &\text{subject to}\ \mathbb{P}\Big[G_i(z) \le \eta_{ik}\Big] \le v_{ik}, \quad k = 1, \dots, k_i, \quad i = 1, \dots, m, \\ &\qquad z \in Z. \end{aligned}$$

The necessary optimality conditions for this relaxation yield a solution of the optimality conditions of the original problem

Sufficient Conditions of Optimality

Assumptions:

(i) for P-almost all $\omega \in \Omega$ the function $[H(\cdot)](\omega)$ is concave on $\mathcal{Z}$

(ii) Each function G_i has the structure:

$$G_i(z) = g_i(z, V_i), \quad i = 1..m,$$

where $g_i : Z \times \mathbb{R}^{s_i} \to \mathbb{R}$, and $V_i : \Omega \to \mathbb{R}^{s_i}$ is a random vector.

(iii) Each function g_i is a quasi-concave function.

(iv) Each vector V_i has an α_i-concave probability density function with $\alpha_i \geq -1/s_i$.

Theorem 6: Assume that a point $\hat{z}$ is feasible. Suppose that there exists a function $\hat{u} \in \mathcal{U}_1^m$, $\hat{u} \neq 0$, such that conditions (10)–(11) are satisfied. Then $\hat{z}$ is an optimal solution of problem (FSD).

Definition A function $\varphi : \mathbb{R}^s \to \mathbb{R}$ is r-concave ($r \in [-\infty, \infty]$), if

$$\varphi(\lambda x + (1-\lambda)y) \geq m_r(\varphi(x), \varphi(y), \lambda)$$

for all $x, y \in \mathbb{R}^s$ and all $\lambda \in [0,1]$, where

$$m_r(a,b,\lambda) = \begin{cases} a^\lambda b^{1-\lambda} & \text{if } r = 0, \\ \max\{a,b\} & \text{if } r = \infty, \\ \min\{a,b\} & \text{if } r = -\infty, \\ (\lambda a^r + (1-\lambda)b^r)^{1/r} & \text{otherwise.} \end{cases}$$

for $r = -\infty$ we call φ quasi-concave

for $r = 0$, φ is called log-concave.