Optimal Security Inspection with a Single-server Queue

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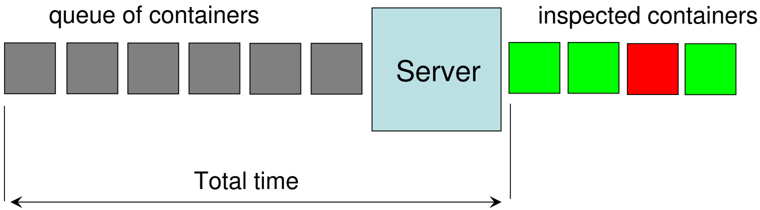
Otline

- Motivation
- Formulation of security inspection problem
- Approach of Riemann boundary-value problems for analytic functions to Wiener-Hopf integral equation
- Optimization problem for distribution parameters
- Examples with frequently used distribution
- Conclusions

Motivation

1. **Passenger screening at airports:**
   Randomized security inspection versus deterministic profiling systems, e.g., Computer Assisted Passenger Prescreening System (CAPPS), introduced in 1999

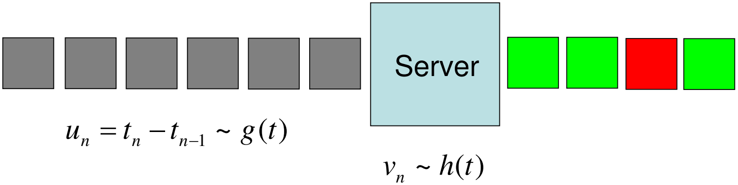
2. **Container examination at ports**



Queueing Process

1. $u_n = t_n - t_{n-1} =$ interarrival time of the n^{th} entity,
   $u_n \sim g(t) \begin{cases} = 0, & t < 0 \\ \geq 0, & t \geq 0 \end{cases}$
2. $v_n =$ inspection (service) time for the n^{th} entity,
   $v_n \sim h(t) \begin{cases} = 0, & t < 0 \\ \geq 0, & t \geq 0 \end{cases}$
3. $w_n =$ queue waiting time for the n^{th} entity to be inspected.

$$w_{n+1} = \max\{w_n + v_n - u_{n+1}, 0\}, \quad n \in \mathbb{N},$$



Steady State of Queueing Process

- $u_n - v_{n+1} \sim k(t) = \int_{-\infty}^{\infty} h(t+\tau)\, g(\tau)\, d\tau$

$$f_{n+1}(t) = \int_0^{\infty} k(t-\tau)\, f_n(\tau)\, d\tau + F_n(0)\, k(t), \qquad t \geq 0$$

- For $n \to \infty$, $f_n(t) \to f(t)$ and $F_n(0) \to F(0) = 1 - \int_0^{\infty} f(t)\, dt$

- **Wiener-Hopf integral equation**

$$f(t) - \int_0^{\infty} k(t-\tau) f(\tau)\, d\tau = F(0)\, k(t), \qquad t \geq 0$$

- System stability condition $E[v] < E[u]$

Performance Functionals

1. $\mathcal{H}$ = set of nonnegative random variables

$$v + c \in \mathcal{H} \qquad \forall\, v \in \mathcal{H} \text{ and } \forall\, c \in \mathbb{R}^+$$

2. $\mathcal{F}(v)$ = performance functional

$$\mathcal{F}(v) \le \mathcal{F}(v + c) \qquad \forall\, v \in \mathcal{H} \text{ and } \forall\, c \in \mathbb{R}^+$$

- The percentage of entities subjected to the security inspection with time greater than a specified threshold T_1

$$\mathcal{F}(v) = \mathrm{Prob}\{v \ge T_1\} = \int_{T_1}^{\infty} h(t)\, dt$$

- The minimal inspection time specified for $\alpha * 100\%$ percentage of all entities

$$\mathcal{F}(v) = \inf\{t \,|\, \mathrm{Prob}\{v \ge t\} \ge \alpha\}$$

Problem Formulation

Maximize the performance functional subject to a constraint on the total time spent in the system

$$\max_{v \in \mathcal{H}} \mathcal{F}(v)$$

s.t.

$$f(t) - \int_0^\infty k(t-\tau)\, f(\tau)\, d\tau = F(0)\, k(t), \quad t \geq 0,$$

$$E[v + w] \leq T, \quad T \in \mathbb{R}^+ \quad \text{(expected time in the system)}$$

$$E[v] < E[u] \quad \text{(stability condition)}$$

(*calculus of variations problem with isoperimetric constraints*)

Exponential Interarrival time

$$u \sim \lambda\, \mathrm{e}^{-\lambda t},\ t \geq 0 \quad \Rightarrow \quad k(t) = \lambda \int_{-\infty}^{\infty} h(t+\tau)\, \mathrm{e}^{-\lambda t}\, d\tau$$

$$f(t) - \int_0^{\infty} k(t-\tau)\, f(\tau)\, ds = F(0)\, k(t), \quad t \geq 0$$

Three methods to derive the steady-state waiting-time distribution for a single-server queue:

(a) The method of embedded Markov chains (Cohen);

(b) The method of Riemann-Hilbert boundary-value problems for moment generation functions
(Fayolle and Iasnogorodski, 1979, and further Cohen and Boxma, 1983);

(c) The method of Wiener-Hopf integral equations (Lindley, 1952).

Waiting time for M/G/1/FCFS Queue

Theorem

For interarrival time $u \sim \lambda\, \mathrm{e}^{-\lambda t}$, $t \geq 0$, and arbitrary inspection time, v, the distribution of the steady-state waiting-time, w, has the probability atom at $t = 0$

$$\Pr\{w = 0\} = 1 - \lambda\, E[v],$$

and for $t > 0$, its probability density function, $f(t)$, is given by

$$f(t) = (1 - \lambda\, E[v])\, \frac{\mathrm{i}\lambda}{2\pi} \int\limits_{-\infty}^{\infty} \frac{(1 - H^{+}(s))\, \mathrm{e}^{-\mathrm{i}st}}{s - \mathrm{i}\lambda\, (1 - H^{+}(s))}\, ds, \qquad t > 0,$$

where $H^{+}(s) = \int\limits_{0}^{\infty} h(\tau)\, \mathrm{e}^{\mathrm{i}s\tau}\, d\tau$ and $\mathrm{i} = \sqrt{-1}$.

Sketch of the Proof

$$f(t) - \int_0^\infty k(t-\tau) f(\tau)\, d\tau = F(0)\, k(t) + f_-(t), \qquad t \in \mathbb{R}.$$

Wiener-Hopf integral equation → Fourier integral transform

$$\Phi^+(s) = \int_0^\infty f(t)\, \mathrm{e}^{\mathrm{i}st} dt, \qquad \Phi^-(s) = \int_{-\infty}^0 f_-(t)\, \mathrm{e}^{\mathrm{i}st} dt, \qquad s \in \mathbb{R},$$

$$K(s) = \int_{-\infty}^\infty k(t)\, \mathrm{e}^{\mathrm{i}st} dt = -\frac{\mathrm{i}\lambda}{s - \mathrm{i}\lambda} \int_0^\infty h(\tau)\, \mathrm{e}^{\mathrm{i}s\tau} d\tau = -\frac{\mathrm{i}\lambda}{s - \mathrm{i}\lambda} H^+(s)$$

Riemann-boundary value problem (RBVP) for piece-wise analytic function $\Phi(z)$ in the complex plane with the branch cut along $\mathbb{R}$:

$$\Phi^+(s)\, [1 - K(s)] = F(0) K(s) + \Phi^-(s), \qquad s \in \mathbb{R}$$

or

$$\Phi^+(s) = \frac{s - \mathrm{i}\lambda}{s - \mathrm{i}\lambda\,(1 - H^+(s))}\, \Phi^-(s) - \frac{\mathrm{i}\lambda\, F(0)\, H^+(s)}{s - \mathrm{i}\lambda\,(1 - H^+(s))}, \qquad s \in \mathbb{R}.$$

Simplest Riemann-boundary Value Problem

RBVP on $\mathbb{R}$ (applications in fluid mechanics, elasticity, etc.)

$$\Phi^+(s) = G(s)\,\Phi^-(s) + g(s), \qquad s \in \mathbb{R}.$$

Factorization:

$$\frac{X^+(s)}{X^-(s)} = G(s) \quad \Longrightarrow \quad \ln X^+(s) - \ln X^-(s) = \ln G(s)$$

Sokhotski formulas:

$$X^\pm(s) = \pm\frac{1}{2}\ln G(s) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\ln G(t)}{t-s}\,dt, \qquad s \in \mathbb{R}$$

$$\frac{\Phi^+(s)}{X^+(s)} = \frac{\Phi^-(s)}{X^-(s)} + \frac{g(s)}{X^+(s)}, \qquad s \in \mathbb{R}.$$

$$\frac{\Phi^\pm(s)}{X^\pm(s)} = \pm\frac{g(s)}{2X^+(s)} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(t)}{X^+(t)}\,\frac{dt}{t-s}, \qquad s \in \mathbb{R}$$

Sketch of the Proof (cont'd)

In this case, $G(s) = \frac{s - i\lambda}{s - i\lambda\,(1 - H^{+}(s))}$ with $E[v] = \int_0^\infty t\,h(t)dt < \frac{1}{\lambda}$

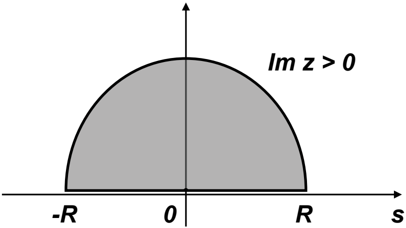
$$X^{+}(z) = \frac{1}{1 - i\lambda\left(\frac{1 - H^{+}(z)}{z}\right)}, \qquad \operatorname{Im} z \geq 0,$$

$$X^{-}(z) = \frac{1}{z - i\lambda}, \qquad \operatorname{Im} z \leq 0,$$

Rouche's theorem: if $|A(z)| > |B(z)|$ for ∂D, then $A(z)$ and $A(z) + B(z)$ have the same number of zeros in $\mathcal{D}$.

$A(z) = 1$ and $B(z) = \frac{1 - H^{+}(z)}{z}$.

$$1 > \lim_{s \to 0} \frac{1 - H^{+}(s)}{s} = \lambda E[v]$$



Sketch of the Proof (cont'd)

RBVP on the infinite contour

$$\Phi^{+}(s) = G(s)\,\Phi^{-}(s) + g(s), \qquad s \in \mathbb{R}.$$

Factorization:

$$G(s) = \frac{X^{+}(s)}{X^{-}(s)} = \frac{\frac{1}{1 - i\lambda\left(\frac{1 - H^{+}(s)}{s}\right)}}{\frac{1}{s - i\lambda}}$$

$$\frac{\Phi^{+}(s)}{X^{+}(s)} - \frac{\Phi^{-}(s)}{X^{-}(s)} = \frac{g(s)}{X^{+}(s)} = \Psi^{+}(s) - \Psi^{-}(s), \qquad s \in \mathbb{R}.$$

$$\frac{\Phi^{+}(s)}{X^{+}(s)} - \Psi^{+}(s) = \frac{\Phi^{-}(s)}{X^{-}(s)} - \Psi^{-}(s) \equiv const$$

The solution to the RBVP takes on the form

$$\Phi^{+}(s) = \frac{i\lambda\, F(0)\ (1 - H^{+}(s))}{s - i\lambda\,(1 - H^{+}(s))}$$

Positiveness of $f(t)$

$B(s) = \frac{1-H^+(s)}{s}$ such that $|B(s)| < 1$.

For $t > 0$:

$$f(t) = (1 - \lambda\, E[v]) \sum_{n=1}^{\infty} \psi_n(t), \qquad \psi_n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B^n(s)\, \mathrm{e}^{-\mathrm{i}st} ds$$

$$\psi_1(t) = \mathrm{Prob}\{v \geq t\}, \quad t > 0$$

$$\psi_n(t) = \underbrace{\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty}}_{n-1} \psi_1\left(t - \sum_{k=1}^{n-1} \tau_k\right) \prod_{k=1}^{n-1} \psi_1(\tau_k)\, d\tau_1 \dots d\tau_{n-1} > 0$$

$$\psi_n(t) \leq \lambda\, (\lambda E[v])^{n-1}$$

Optimal Security Inspection for M/G/1 Queue

Pollaczek-Khinchin formulas (corollary from the theorem):

$$E[w] = \frac{\lambda E[v^2]}{2(1-\lambda E[v])}, \quad \mathrm{var}[w] = \frac{\lambda E[v^3]}{3(1-\lambda E[v])} + \left(\frac{\lambda E[v^2]}{2(1-\lambda E[v])}\right)^2$$

Theorem (Optimal security inspection for M/G/1 queue)

The optimal security problem for M/G/1 *queue reduces to*

$$\max_{v \in \mathcal{H}} \mathcal{F}(v)$$

$$\text{s.t.} \quad E[v] \le \frac{1}{\lambda} + T - \sqrt{\frac{1}{\lambda^2} + T^2 + \mathrm{var}[v]}, \quad \mathrm{var}[v] \in \left[0, \frac{2T}{\lambda}\right],$$

where $\mathrm{var}[v]$ *is the variance of the inspection time* v*.*

Shifted-exponential Inspection Time

Inspection time, v, follows shifted-exponential distribution with the shift a and rate γ

$$h(t) = \begin{cases} 0, & t < a \\ \gamma\, \mathrm{e}^{\gamma\,(a-t)}, & t \geq a \end{cases}$$

System stability condition reduces to $a + \frac{1}{\gamma} < \frac{1}{\lambda}$, and the steady-state waiting-time, w, is determined by

$$\Pr\{w = 0\} = 1 - \lambda\left(a + \frac{1}{\gamma}\right),$$

$$f(t) = \frac{\lambda}{2\pi}\left(1 - \lambda\left(a + \frac{1}{\gamma}\right)\right)\int_{-\infty}^{+\infty} \frac{\left(\mathrm{i} + \frac{\gamma}{s}\left(\mathrm{e}^{\mathrm{i}as} - 1\right)\right)\mathrm{e}^{-\mathrm{i}st}}{s + \mathrm{i}(\gamma - \lambda) - \frac{\lambda\gamma}{s}\left(\mathrm{e}^{\mathrm{i}as} - 1\right)}\, ds$$

for $t \geq 0$.

Shifted-exp. Inspection Time: Waiting Time

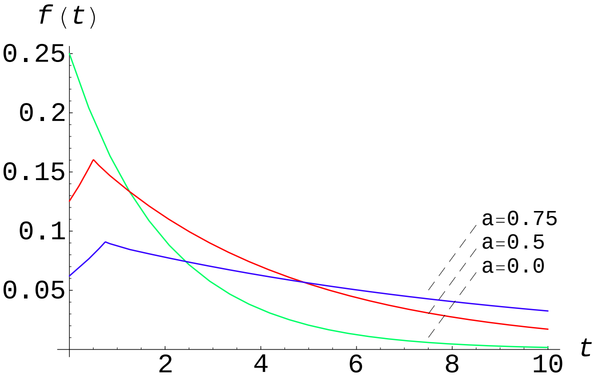


Figure: Density function $f(t)$ for $\lambda = 0.5$ and shifted-exponential inspection time with $\gamma = 1$, and $a = 0.0, 0.5, 0.75$

Shifted-exp. Inspection Time: Waiting Time

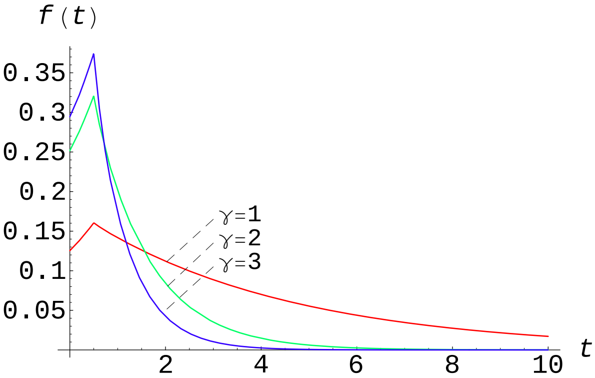


Figure: Density function $f(t)$ for $\lambda = 0.5$ and shifted-exponential inspection time with $a = 0.5$, and $\gamma = 1, 2, 3$

Shifted-exp. Inspection Time: Waiting Time

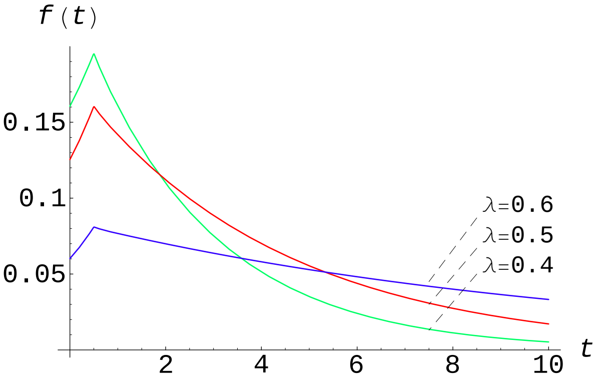


Figure: Density function $f(t)$ for $\lambda = 0.4$, 0.5, 0.6 and shifted-exponential inspection time with $a = 0.5$ and $\gamma = 1$

Shifted-exp. Inspection Time: Optimal Parameters

Performance functional: $\mathcal{F}(v) = \mathrm{Prob}\{v \geq T_1\} = \int_{T_1}^{\infty} h(t)\, dt$

Let $d_1 = T + \frac{1}{\lambda} - \sqrt{\frac{1}{\lambda^2} + T^2}$ and $d_2 = \frac{T(1+\lambda T)}{1+\lambda T+\lambda^2 T^2}$

1. if $T_1 \geq d_2$, then $a = 0$ and $\gamma = \lambda + \frac{1}{T}$
2. If $d_1 \leq T_1 \leq d_2$, then

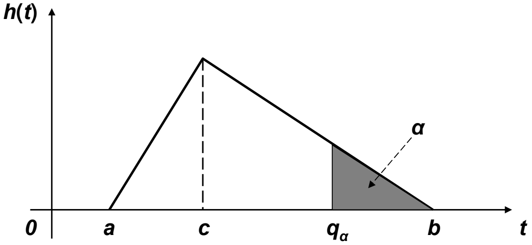
$$a = \frac{\frac{2T}{\lambda} - T_1\left(\frac{1}{\lambda} + T\right) - \sqrt{\left(\frac{1}{\lambda^2} + T^2\right)\left(\frac{2(T_1 - T)}{\lambda} + 2T_1 T - T_1^2\right)}}{\frac{1}{\lambda} + T - T_1}$$

$$\gamma = \frac{\frac{1}{\lambda} + T - T_1}{\sqrt{\left(\frac{1}{\lambda^2} + T^2\right)\left(\frac{2(T_1 - T)}{\lambda} + 2T_1 T - T_1^2\right)}}$$

3. If $T_1 \leq d_1$, then $a = d_1$ and $\gamma \to \infty$.

Triangular Inspection Time

Inspection time, v, follows triangular distribution with parameters a, b and c



System stability condition: $a+b+c<\frac{3}{\lambda}$, and the function H^+ takes on the form

$$H^+(s)=-\frac{2\left((b-c)\,\mathrm{e}^{ias}+(c-a)\,\mathrm{e}^{ibs}-(b-a)\,\mathrm{e}^{ics}\right)}{(b-a)(c-a)(b-c)\,s^2}.$$

Triangular Inspection Time: Waiting Time

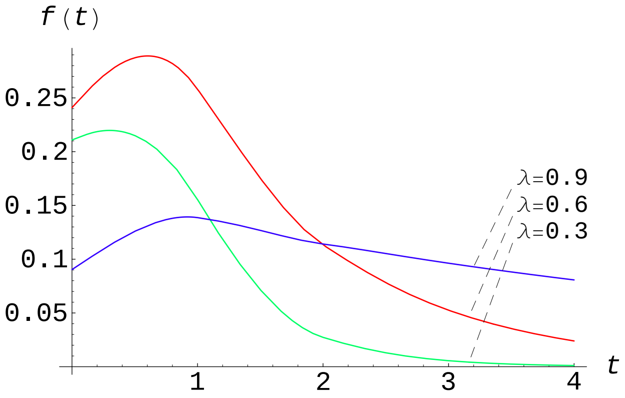


Figure: Density function $f(t)$ for $\lambda = 0.3$, 0.6, 0.9 and triangular inspection time with $a = 0$, $c = 1$, $b = 2$

Triangular Inspection Time: Waiting Time

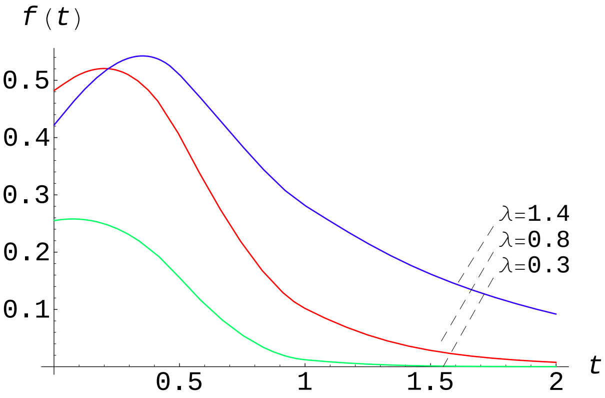


Figure: Density function $f(t)$ for $\lambda = 0.3$, 0.8, 1.4 and triangular inspection time with $a = 0$, $c = 0.5$, $b = 1$

Triangular Inspection Time: Optimal Security problem

Performance functional: $\mathcal{F}(v) = \inf\{t \,|\, \mathrm{Prob}\{v \geq t\} \geq \alpha\}$

$$\mathcal{F}(a,b,c,\alpha) = \begin{cases} b - \sqrt{\alpha(b-c)(b-a)}, & \alpha \leq \frac{b-c}{b-a}, \\ a + \sqrt{(1-\alpha)(c-a)(b-a)}, & \alpha > \frac{b-c}{b-a}. \end{cases}$$

Optimal security problem: parameter optimization

$$F(\alpha,\lambda) = \max_{b \geq c \geq a \geq 0} \quad \mathcal{F}(a,b,c,\alpha)$$

$$\text{s.t.}$$

$$\frac{1}{\lambda} + T - \sqrt{\frac{1}{\lambda^2} + T^2 + \frac{1}{18}(a^2+b^2+c^2-ab-ac-bc)} \leq \frac{1}{3}(a+b+c)$$

Triangular Inspection Time: Numerical Analysis

$$F(\alpha,\lambda) = \max_{a,b,c} \inf \{ t \,|\, \mathrm{Prob}\{ v \geq t \} \geq \alpha \} \text{ s.t. constraints}$$

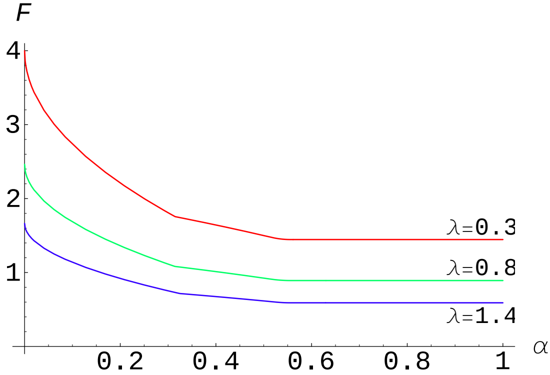


Figure: Optimal value of the performance functional as a function of α for $\lambda = 0.3, 0.8, 1.4$, and $T = 2$

Conclusions

1. Formalization of optimal security-inspection problem
2. Derivation of the Pollachek-Khinchin formulas for the $M/G/1$ queue in the framework of RBVP for analytic functions
3. Analytical treatment of the optimal security-inspection problem for the $M/G/1$ queue
4. Optimal security inspection for the $M/G/1$ queue and shifted-exponential and triangular inspection time

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