Risk Adjusted Probabilities in Portfolio Optimization with Coherent Measures of Risk

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Portfolio Optimization Problem

- We consider **n-assets** whose returns $\mathbf{R^j}$ are integrable random variables, on some probability space $(\mathbf{\Omega}, \mathbf{F}, \mathbf{P})$.
- The vector $\mathbf{z} \in \mathbf{R^n}$ represents our asset allocation, with each $\mathbf{z_j}$ equal to the fraction of C invested in asset j
- The set of possible asset allocations

$$\mathbf{Z} = \{z \in R^n : z_1 + .. + z_n = C, z_j \geq 0, j = 1..n\} \tag{1}$$

- The return of the portfolio is given by $\mathbf{R^T z}$
- Abstract risk-averse portfolio optimization problem.

Coherent risk measures

- $(\Omega, F, P) \rightarrow$ probability space
- $X : \Omega \rightarrow R \rightarrow$ uncertain outcome
- $K \rightarrow$ vector space of possible outcomes $\qquad K \in \mathcal{L}_1(\Omega, F, P)$

Definition: A coherent measure of risk is a functional $\rho : K \rightarrow R$ satisfying the following axioms:

- **Convexity**: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$, for all $X, Y \in K$ and all $\alpha \in [0, 1]$;
- **Monotonicity**: If$X, Y \in K,\ and\ X(\omega) \leq Y(\omega)\ \forall\omega \in K$, then $\rho(X) \geq \rho(Y)$
- **Translational Equivariance**: If $\alpha \in R$ and $X \in K$, then $\rho(X + \alpha) = \rho(X) - \alpha$
- **Positive Homogeneity**: If $t > 0$ and $X \in K$, then $\rho(tX) = t\rho(X)$

Mean-risk model

$$\rho(X) = -E[X] + \lambda r[X] \tag{2}$$

for certain $r[.]$ and λ

Semideviation

$$r[X] = \sigma_p(X) = E[(E[X] - X)_+^p]^{\frac{1}{p}} \tag{3}$$

Weighted Mean Deviation from Quantile

$$r_\alpha(X) = \min_{\eta \in R} E[\max(\frac{1-\alpha}{\alpha}(\eta - X), X - \eta)], \qquad \alpha \in (0,1) \tag{4}$$

Risk-Averse Portfolio Optimization Problem

$$\min_{z \in Z} \rho(R^T z) \tag{5}$$

Outline

- Examine this problem when ρ is a coherent risk measure
- Derive an equivalent dual problem
- Dual Optimality conditions
- Derivation of risk - adjusted probability measures
- Construction of risk-adjusted probability measures

- Let M denote the space of regular, countably additive signed measures,**u** on (Ω, F), absolutely continuous w.r.t. P. The densities $\frac{du}{dP} \in L_\infty(\Omega, F, P)$
- Let **P** denote the set of probability measures in M

Representation Theorem: If ρ is a lower semicontinuous coherent measure of risk, then there exists a convex and weakly* closed set $A \subset \mathbf{P}$ such that

$$\rho(X) = \sup_{u \in A}(-\int_\Omega X(\omega)u(d\omega), \qquad X \in K \tag{6}$$

The mean-risk models with semideviations and deviations from quantile satisfy assumptions of theorem. The portfolio optimization problem can be written as

$$\min_{z \in Z} \sup_{u \in A}(-\int_\Omega R^T(\omega)zu(d\omega)). \tag{7}$$

Optimality Theorem

Theorem: Suppose ρ is a continuous coherent measure of risk. A point $\overline{z}$ is an optimal solution of problem (6) $\Leftrightarrow$ $\exists$ a convex and weakly* closed set $\overline{A} \subset A$ such that $\forall$ probability measures $\overline{u} \in \overline{A}$ the point $\overline{z}$ is also a solution of the problem

$$\max_{z \in Z} \int_{\Omega} R^T(\omega) z \overline{u}(d\omega). \tag{8}$$

Furthermore, the set $\overline{A}$ is the set of solutions to the dual problem

$$\min_{u \in A} \max_{z \in Z} \int_{\Omega} R^T(\omega) z u(d\omega). \tag{9}$$

Proof

$F(z, u)$ function defined on $Z \times A$,

$$F(z, u) = \int_\Omega R^T(\omega) z u(d\omega) \tag{10}$$

The optimal value of $F(z, u)$ occurs at saddle point $(\overline{z}, \overline{u})$,

$$F(z, \overline{u}) \leq F(\overline{z}, \overline{u}) \leq F(\overline{z}, u) \tag{11}$$

The optimal portfolio $\overline{z}$ optimizes expected return w.r.t the probability measure $\overline{u}$.

u is the risk-adjusted probability measure

Discrete case

- Assume our probability space is finite, with N elementary events $\boldsymbol{\omega_1}, .., \boldsymbol{\omega_N}$, and corresponding probabilities $\mathbf{p_1}, .., \mathbf{p_N}$
- The **Portfolio return** is given by $\mathbf{X_z} = \sum_{\mathbf{j=1}}^{\mathbf{N}} \mathbf{z_j R_j}$
- Set of possible portfolios is given by the simplex

$\mathbf{Z} = \{\mathbf{z} \in \mathbf{R^n} : \mathbf{z_1} + \mathbf{z_2} + ... + \mathbf{z_n} = \mathbf{1}, \mathbf{z_j} \geq \mathbf{0}, \mathbf{j} = \mathbf{1...n}\}$

The portfolio problem in this case can be written as

$$\max_{z \in Z} \min_{u \in A} \langle u, R^T z \rangle \tag{12}$$

The dual problem has form

$$\min_{u \in A} \max_{z \in Z} \langle u, R^T z \rangle \tag{13}$$

This can be interpreted as a matrix game, with payoff matrix $\boldsymbol{R^T}$ and mixed strategies of players represented by portfolio allocation $\mathbf{z}$ and measure $\mathbf{u}$

The mean-semideviation model

$$\min_{z \in Z} -E[X] + \lambda \sigma_1(X) \tag{14}$$

where σ_1 is the semideviation risk measure, with $p = 1$ and $\lambda \in [0, 1]$ some fixed constant.

$$\sigma_1 = E \max(E[X] - X, 0) \tag{15}$$

Discrete case

$$\min_{z \in Z} -\langle p, R^T z \rangle + \lambda \sum_{i=1}^{N} p_i \max(\langle p, R^T z \rangle - \langle r_i, z \rangle, 0) \tag{16}$$

where $r_i \in R^n$ represents vector of asset returns in event i.

Lagrangian

Equivalent formulation of (12)

$$\begin{aligned} \text{Minimize } & -\langle p, R^T z\rangle + \lambda\langle p, s\rangle \\ \text{s.t. } & s_i \geq \langle p, R^T z\rangle - \langle r_i, z\rangle \qquad (1) \\ \text{s.t. } & s \geq 0 \qquad z \in Z \end{aligned}$$

The Lagrange Function

$$L(z, s, \xi) = (\langle \xi, 1\rangle - 1)\langle p, R^T z\rangle - \langle \xi, R^T z\rangle + \langle \lambda p - \xi, s\rangle \tag{17}$$

where ξ_i denote lagrange multipliers associated with constraints in (1)

Theorem: Suppose $\rho(X) = -E(X) + \lambda\sigma_1(X)$ with $\lambda \in [0, 1]$. A vector $\overline{z}$ and a measure $\overline{u}$ constitute a saddle point of game (9) $\Leftrightarrow$ the vector z is a solution of problem (1) and

$$\overline{\mathbf{u}} = (\mathbf{1} - \langle\overline{\xi}, \mathbf{1}\rangle\mathbf{p} + \overline{\xi}, \tag{18}$$

where $\overline{\xi}$ are the lagrange multipliers associated with constraints in (1).

The convex programming dual problem coincides with game-theoretic dual

$$\max_{u \in A} \min_{z \in Z} \langle -u, R^T z\rangle \tag{19}$$

where the set of probability measures A is given by

$$A = -\partial\rho[0] = \{(1 - \lambda\langle g, 1\rangle)p + \lambda g : |g_i| \leq p_i, i = 1, .., N\} \tag{20}$$

Quantile

The mean-deviation from quantile model

$$\rho(X) = -E[X] + \lambda r_\alpha[X] \tag{21}$$

where $\lambda \in [0, 1]$ is fixed. and

$$r_\alpha(X) = \min_{\eta \in R} E[\max(\frac{1-\alpha}{\alpha}(\eta - X), X - \eta)], \qquad \alpha \in (0, 1) \tag{22}$$

Discrete Case

$$\rho(X) = -\langle p, X \rangle + \lambda \min_{\eta \in R} \sum_{i=1}^{N} p_i \max(\frac{1-\alpha}{\alpha}(\eta - x_i), x_i - \eta). \tag{23}$$

Lagrange function

Equivalent formulation of (18)

$$\begin{aligned}
\text{Minimize } & -\langle p, R^T z\rangle + \lambda \min_{\eta\in R} \sum_{i=1}^{N} p_i\left(\frac{1-\alpha}{\alpha} v_i + u_i\right) \\
\text{s.t. } & \langle r_i, z\rangle - \eta = u_i - v_i, \qquad i = 1, .., N, (1) \\
\text{s.t. } & u_i, v_i \geq 0, i = 1, .., N, \\
\text{s.t. } & \eta \in R
\end{aligned}$$

The Lagrange Function

$$\begin{aligned}
& L(z, \eta, u, v, \xi) \\
& \quad = -\langle p - \xi, R^T z\rangle - \eta\langle \xi, 1\rangle + \langle \frac{\lambda(1-\alpha)}{\alpha} p + \xi, v\rangle + \langle \lambda p - \xi, u\rangle
\end{aligned}$$

Theorem

Theorem: Suppose $\rho(X) = -E[X] + \lambda r_\alpha(X)$ with $\lambda \in [0,1]$. A vector $\overline{z}$ and a vector $\overline{u}$ constitute a saddle point of the game (9) $\Leftrightarrow$ the vector z is a solution of problem (1) and

$$\overline{\mathbf{u}} = \mathbf{p} - \overline{\xi}, \quad (24)$$

where $\overline{\xi}$ are the lagrange multipliers.
The convex programming dual problem coincides with game-theoretic dual

$$\max_{u \in A} \min_{z \in Z} \langle -u, R^T z \rangle \quad (25)$$

where the set of probability measures A is given by

$$A = -\partial\rho[0] = \{(1-\lambda)p + \lambda g : 0 \le g_i \le \frac{p_i}{\alpha}, i = 1, .., N, \sum_{i=1}^{N} g_i = 1\} \quad (26)$$

Kusuoka Theorem: For every lower semicontinuous law invariant coherent risk measure ρ on $L_\infty(\Omega, F, 1)$, there exists a convex set N of probability measures on [0,1] such that

$$\rho(X) = \sup_{v \in N} \int_0^1 AVaR_\alpha[X] v(d\alpha). \tag{27}$$

Using the the following identity for $AVaR_\alpha(X)$,

$$AVaR_\alpha(X) = -\frac{1}{\alpha} \int_0^\alpha q_t(X) dt = -E[X] + r_\alpha[X] \tag{28}$$

we can give an equivalent formulation

$$\rho(X) = -E[X] + \sup_{v \in N} \int_0^1 r_\alpha[X] v(d\alpha). \tag{29}$$

where $r_\alpha(X)$ is the weighted mean deviation from quantile.

Kusuoka model discrete case

Every coherent law invariant risk measure corresponds to a mean-risk model, with the risk functional

$$\kappa_N[X] = \sup_{v \in N} \int_0^1 r_\alpha[X] v(d\alpha) \tag{30}$$

In the discrete case, the risk-functional (29) takes the form

$$\kappa_N[X] = \sup_{v \in N} \sum_{k=1}^{N} v_k r_{\alpha_k}[X], \tag{31}$$

with $\alpha_k = k/N$.

Mean-Risk Portfolio Optimization

$$\text{Minimize}_{z \in Z} - E[X R^T z] + \lambda \kappa_N [R^T z] \tag{32}$$

Portfolio Optimization problem for spectral measure of risk

$$\text{Minimize } -\langle p, R^T z\rangle + \lambda \sum_{k=1}^{N} v_k \sum_{i=1}^{N} p_i \left(\frac{1-\alpha_k}{\alpha_k} v_{ik} + u_{ik}\right) \qquad (A)$$

$$\begin{aligned} \text{s.t. } & \langle r_i, z\rangle - \eta_k = u_{ik} - v_{ik}, \qquad i,k = 1,..,N \\ & u_{ik}, v_{ik} \geq 0, \qquad i,k = 1,..,N, \\ & \eta_k \in R, \qquad k = 1,..,N \\ & z \in Z \end{aligned}$$

We denote by ξ_{ik} the lagrange multipliers associated with constraints in (A). The matrix of lagrangian multipliers is denoted by Ξ.

Theorem Suppose $\rho(X) = -E[X] + \lambda\kappa_N[X]$ with $\lambda \in [0, 1]$ and $N = \{v\}$. A vector $\overline{z}$ and a measure $\overline{u}$ constitute a saddle point of the game(9) $\Leftrightarrow$ the vector z is a solution of problem (A) and

$$\overline{\mathbf{u}} = \mathbf{p} - \overline{\mathbf{\Xi}}\mathbf{1}, \qquad (33)$$

where $\overline{\Xi}$ is a solution of the dual problem (34).

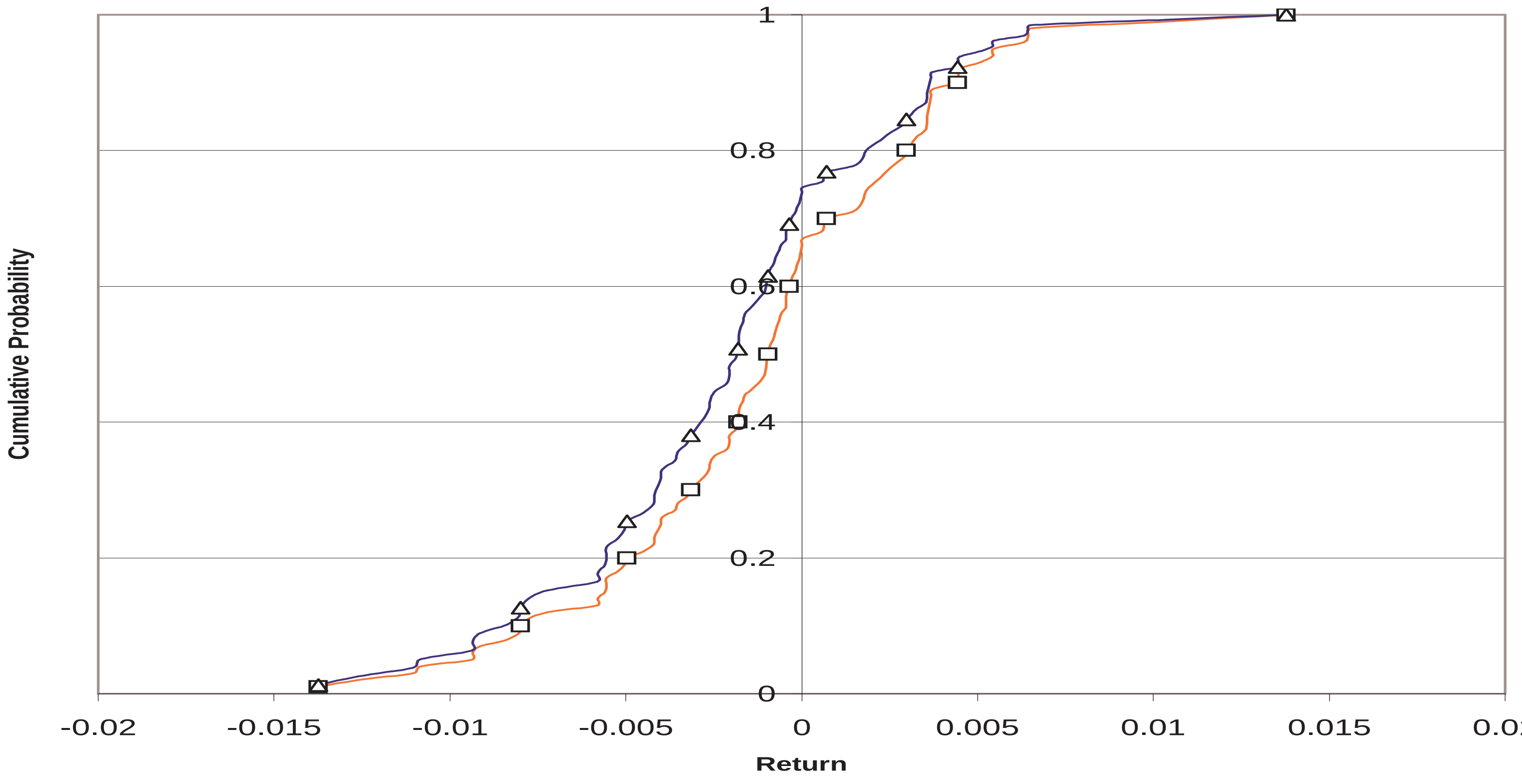

Figure 1: Cumulative distribution curves of market portfolio for the semideviation measure.

Numerical examples

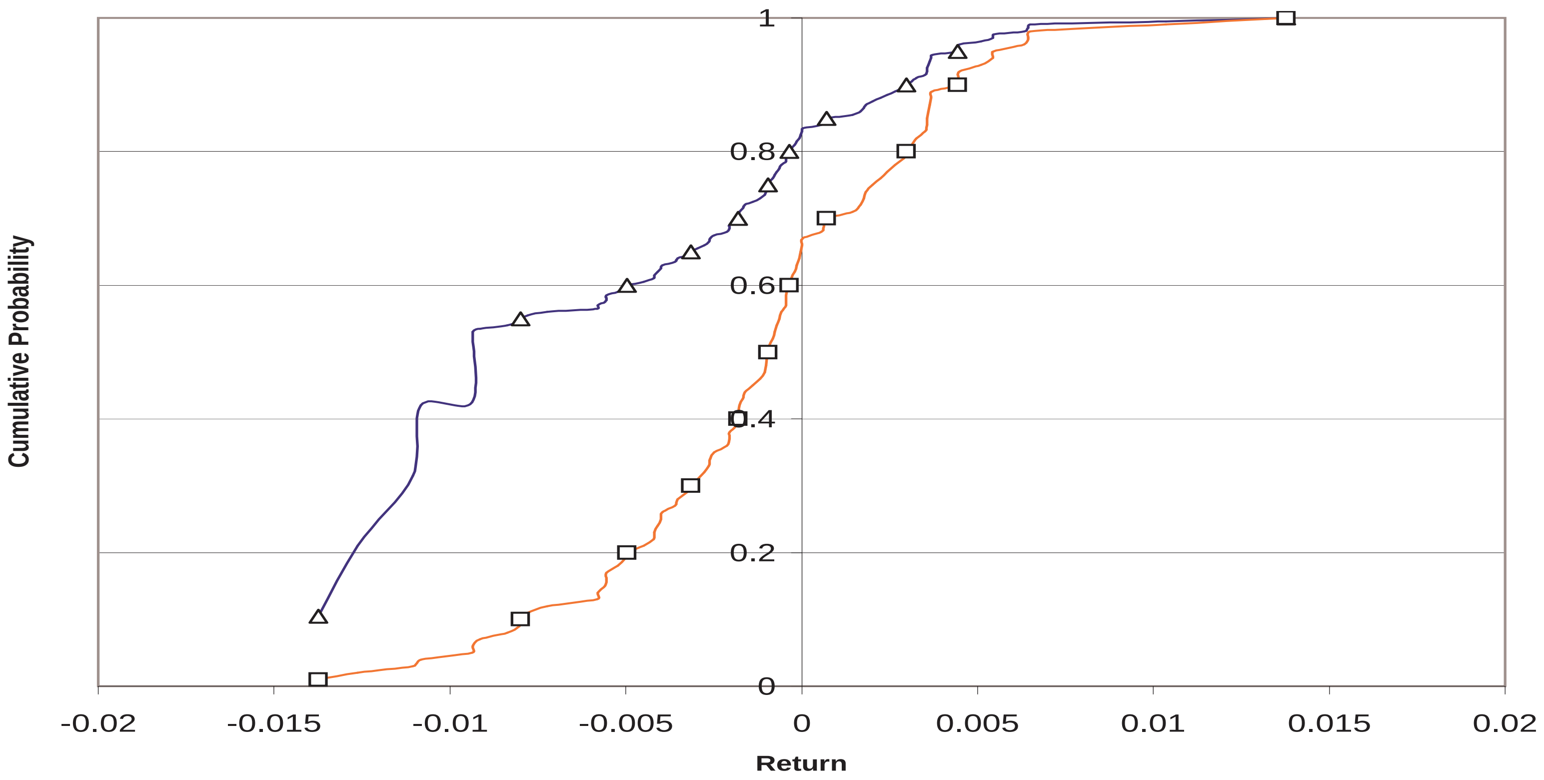

Figure 2: Cumulative distribution curves of market portfolio for the deviation from quantile measure.

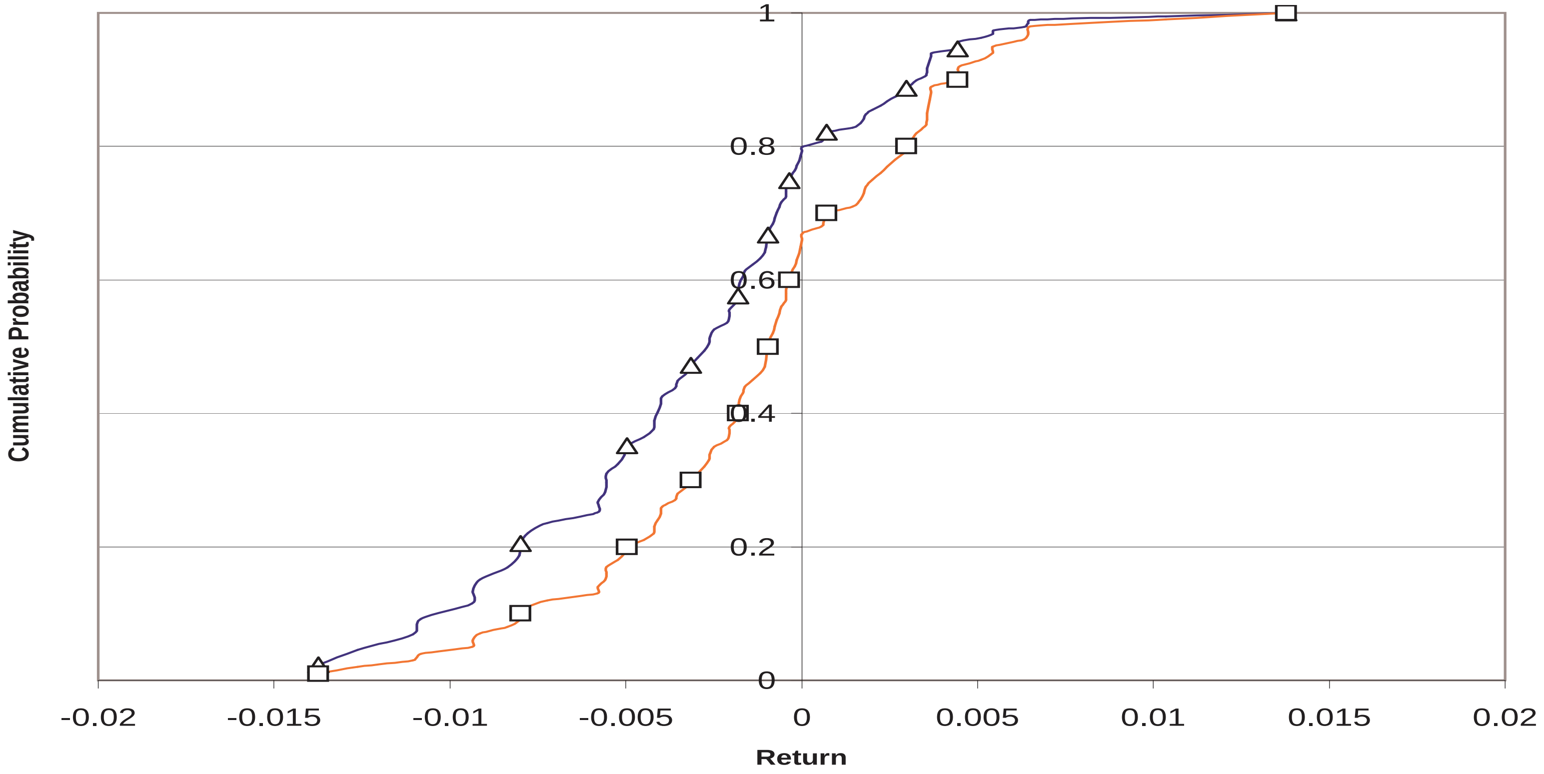

Figure 3: Cumulative distribution curves of market portfolio for the spectral measure.

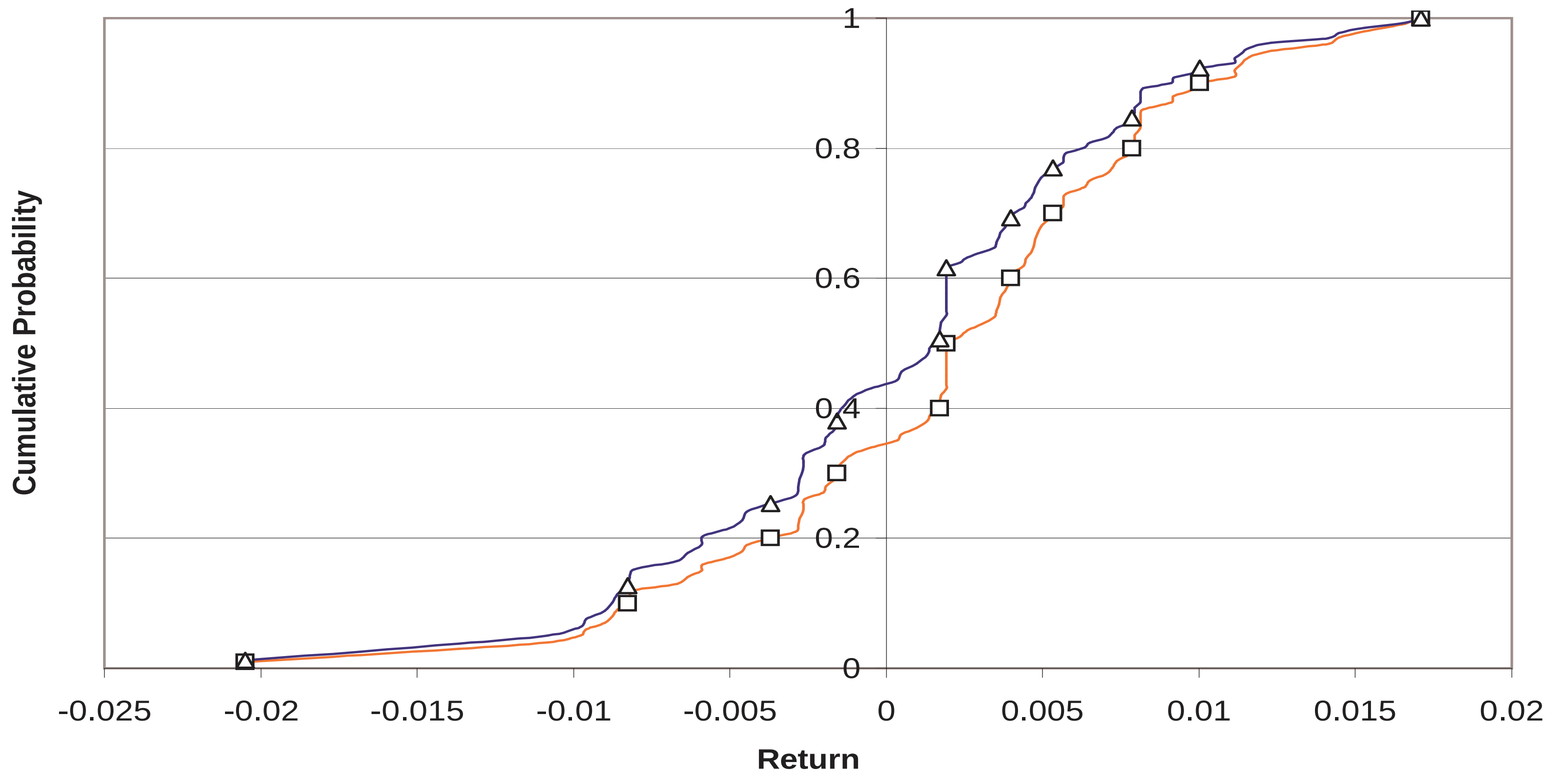

Figure 4: The CDF curves corresponding to optimal portfolio returns for the semideviation measure.

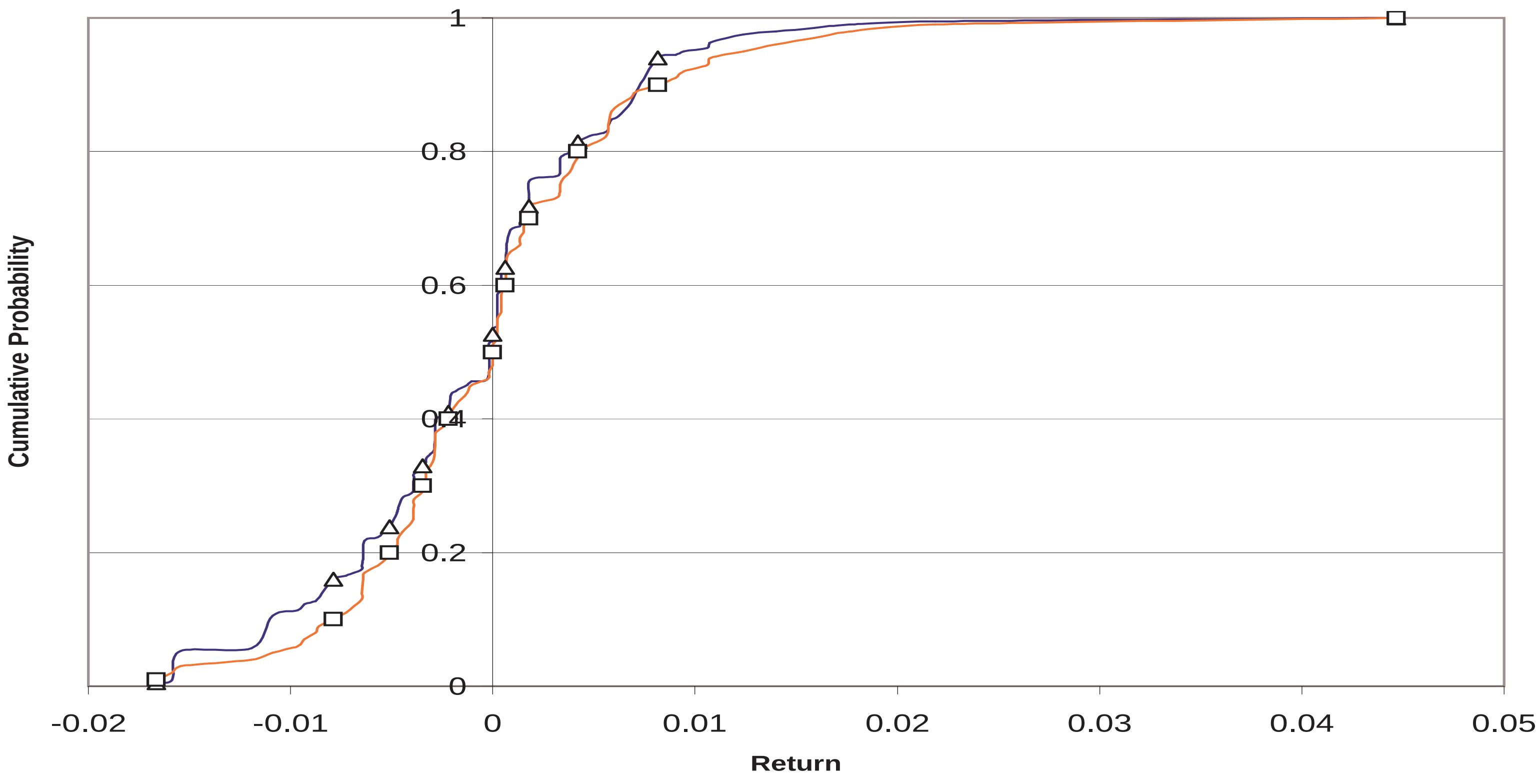

Figure 5: The CDF curves corresponding to optimal portfolio returns for the deviation from quantile measure.

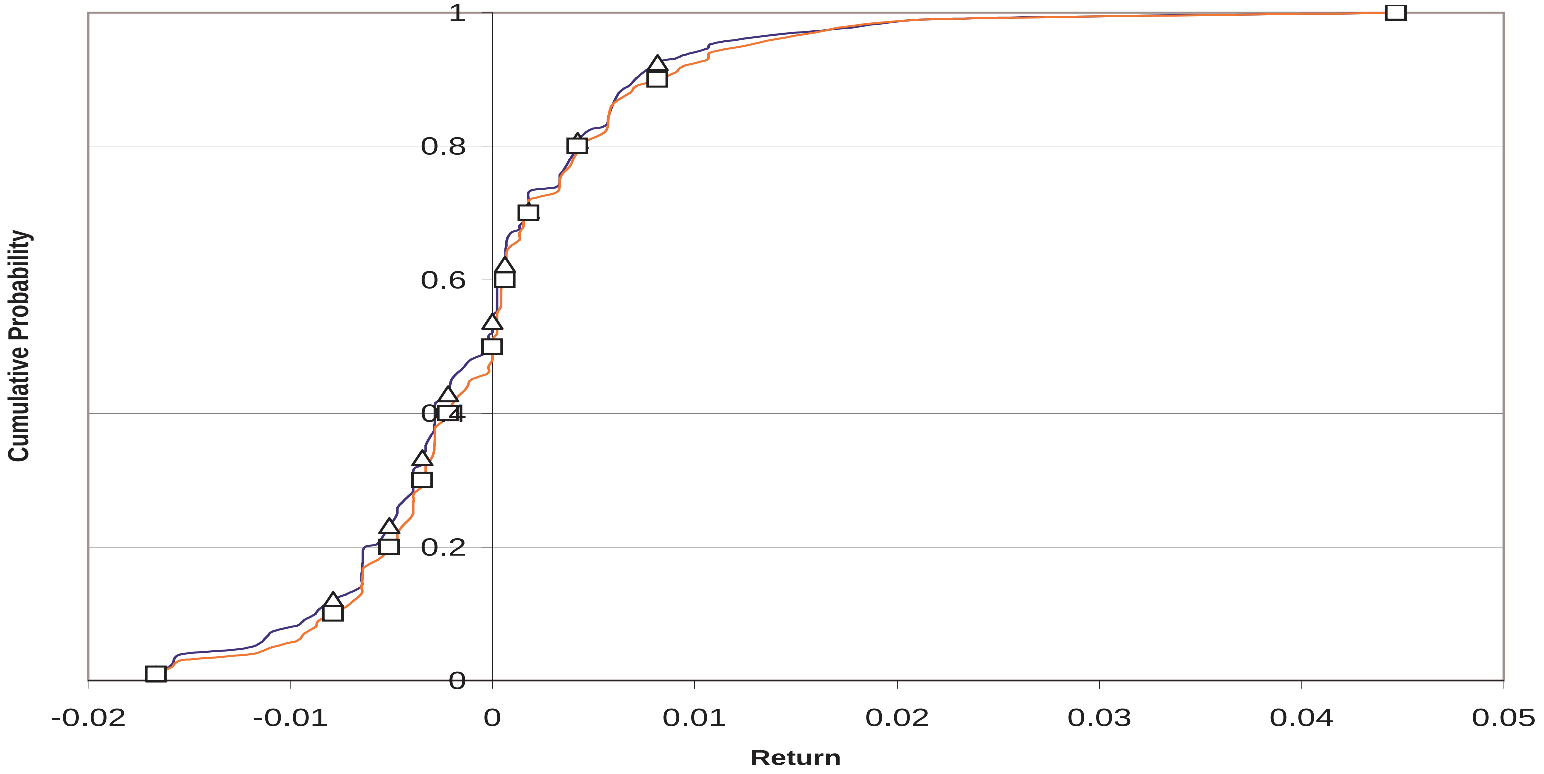

Figure 6: The CDF curves corresponding to optimal portfolio returns for the spectral measure.

Future Research

- Two stage portfolio optimization problem
- Derive Dual problem and optimality conditions for dual
- Derive Risk-Adjusted probability measures for the two stage problem
- Construction of risk-adjusted probability measures
- Extension to multistage portfolio optimization