Stochastic Dominance Constraints Induced by Mixed-Integer Linear Recourse

Rüdiger Schultz
(University of Duisburg-Essen, Germany)

Joint work with:

Frederike Neise

Ralf Gollmer

Uwe Gotzes

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Risk Aversion via Stochastic Dominance

Random mixed-integer linear program:

$$\min \left\{ c^\top x + q^\top y \; : \; Tx + Wy = z(\omega), \; x \in X, \; y \in Y \right\}$$

Information constraint:

select x without anticipation of $z(\omega)$

Two-stage scheme of alternating decision and observation:

$$\text{decide } x \;\mapsto\; \text{observe } z(\omega) \;\mapsto\; \text{decide } y = y(x, \omega)$$

Gives rise to a family of random variables:

$$\min_x \Big\{ c^\top x + \min_y \{ q^\top y \;:\; Wy = z(\omega) - Tx,\; y \in Y \} \;: \; x \in X \Big\}$$

$$= \min_x \{ c^\top x + \Phi(z(\omega) - Tx) \;:\; x \in X \}$$

$$\Big(f(x, z(\omega)) \;:=\; c^\top x + \Phi(z(\omega) - Tx) \Big)_{x \in X}$$

Basic properties of f:

(A1) complete mixed-integer recourse (W rational):

$$W(Y) = \mathbb{R}^s$$

(A2) dual feasibility of LP relaxation (e.g., with $Y = \mathbb{R}^{\bar{m}}_+$):

$$\{u \in \mathbb{R}^s : W^\top u \leq q\} \neq \emptyset$$

Lemma 1:

(A1) + (A2)
$\Longrightarrow f(.,.)$ real-valued and lower semicontinuous (lsc) on $\mathbb{R}^m \times \mathbb{R}^s$.

Examples (in energy optimization):

- Planning a dispersed generation system
  - First stage x:
    What types of and how many generators to install ?
  - Second stage y:
    Operation decisions under stochastic demand, prices, or infeed.

- Day-ahead trading:
  - First stage x: Bids for the day ahead.
  - Second stage y:
    Production and bids on the next day under stochastic bids of foreign utilities.

First Alternative – Find a “best” member in the family!

Mean-risk stochastic (integer) program:

$$\min\{Q_{MR}(x) \ : \ x \in X\}$$

where

$$\begin{aligned} Q_{MR}(x) &:= (\mathbb{E} + \rho \cdot \mathcal{R})\big[f(x, z(\omega))\big] \\ &= \mathbb{E}\big[f(x, z(\omega))\big] \ + \ \rho \cdot \mathcal{R}\big[f(x, z(\omega))\big] \\ &= Q_{\mathbb{E}}(x) + \rho \cdot Q_{\mathcal{R}}(x) \end{aligned}$$

$\mathbb{E}$ – mean value, $\mathcal{R}$ – risk measure, $\rho > 0$ – weight factor.

Summary on Mean-Risk Stochastic Integer Programs:

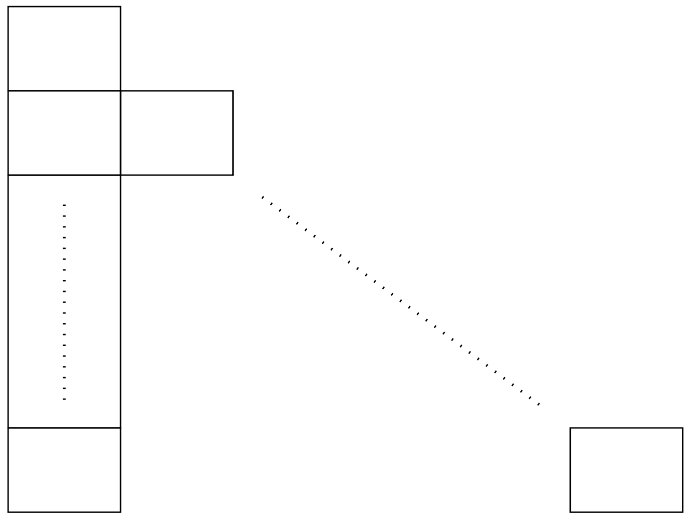
- $\min\{Q_{MR}(x) \; : \; x \in X\}$ is a non-convex optimization problem.

- If $z(\omega)$ follows a discrete probability distribution, then $\min\{Q_{MR}(x) \; : \; x \in X\}$ is a large-scale, block-structured, mixed-integer linear program.

- Example: $\quad Q_{MR}(x) = \mathbb{E}\big[f(x, z(\omega))\big]$

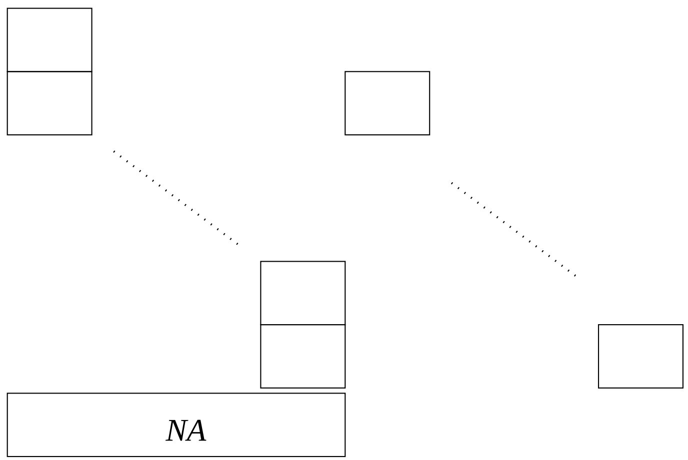
$$\begin{aligned} \min\{c^\top x \; + \; \sum_{j=1}^{J} \pi_j q^\top y_j \quad : \quad & Tx + W y_j \; = \; z_j \\ & x \in X, \; y_j \in Y, \; j = 1, \ldots, J\}. \end{aligned}$$

- Block Structure: No constraint with two second-stage variables y_{j_1} and y_{j_2}.



- Algorithm: Decomposition by Lagrangian Relaxation of Nonanticipativity (NA):

  Introduce copies $x_j, j = 1, \ldots, J$, and add $x_1 = \ldots = x_J$.



Second Alternative – Identify "acceptable" members in the family ! (and optimize over them)

Dominance-constrained stochastic (integer) program:

$$\min\{g(x) \ : \ x \in X, \ \ f(x, z(\omega)) \ \succeq_i \ a(\omega)\}$$

where

- g - an (additional) objective; for instance:

  $g(x) := \|x - \hat{x}\|_1$ deviation from pre-planned schedule $\hat{x}$,

  $g(x) := g^\top x$ displeasure in capacity expansion.

- $a(\omega)$ - a reference or benchmark (cost) random variable,

- $\succeq_i$ - stochastic dominance relation of order $i = 1, 2$.

Stochastic Dominance

(with preference of small outcomes)

First-Order Stochastic Dominance

$f(x,z)$ dominates a with first order ($f(x,z) \succeq_1 a$) if

$$\int_{\mathbb{R}^s} h\big(f(x,z)\big)\,\mu(dz) \;\leq\; \int_{\mathbb{R}} h\big(a\big)\,\nu(da)$$

for all non-decreasing $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that the integrals exist

($\mu \in \mathcal{P}(\mathbb{R}^s)$ – distribution of z, $\nu \in \mathcal{P}(\mathbb{R})$ – distribution of a)

Equivalent Representations:

$$\nu(a \le \eta) \;\; \le \;\; \mu\big(f(x,z) \le \eta\big) \qquad \forall \eta \in I\!R$$

or

$$\nu(a \le \eta) \; + \; \mu\big(M_\eta(x)\big) \;\; \le \;\; 1 \qquad \forall \eta \in I\!R$$

(where $M_\eta(x) := \{z \in I\!R^s \, : \, f(x,z) > \eta\}$ open set, by Lemma 1)

Second-Order Stochastic Dominance

$f(x,z)$ dominates a with second order $(f(x,z) \succeq_2 a)$ if

$$\int_{I\!R^s} h\big(f(x,z)\big)\,\mu(dz) \;\leq\; \int_{I\!R} h\big(a\big)\,\nu(da)$$

for all convex non-decreasing $h : I\!R \longrightarrow I\!R$ such that integrals exist

equivalent:

$$\int_{I\!R^s} [f(x,z)-\eta]_+\,\mu(dz) \;\leq\; \int_{I\!R} [a-\eta]_+\,\nu(da) \;=:\; a_\eta \qquad \forall \eta \in I\!R$$

Structural Analysis

Set-Valued Mapping:

$$C \,:\, \mathcal{P}(I\!R^s) \longrightarrow 2^{I\!R^m},$$

$$C(\mu) := \{x \in I\!R^m \;:\; f(x,z) \succeq_i a,\, x \in X\}, \;\; i = 1, 2$$

$\mathcal{P}(I\!R^s)$ equipped with weak convergence of probability measures

(ν fixed)

Aim: (semi)continuity of C.

Proposition 1 (for $i = 1$):

(A1) + (A2)

$\Longrightarrow C$ closed, i.e., $\mu_n \xrightarrow{w} \mu, x_n \in C(\mu_n), x_n \to x$ imply $x \in C(\mu)$.

Proof:

$$x_n \in C(\mu_n) \qquad \text{iff} \qquad \nu(a \leq \eta) + \mu_n\big(M_\eta(x_n)\big) \leq 1 \quad \forall \eta \in I\!R$$

lsc of f implies:

$$M_\eta(x) \subseteq \liminf_n M_\eta(x_n) \qquad \text{(set-theoretic limes inferior)}$$

lsc of the probability measure implies for all $\eta \in I\!R$ and all n:

$$\mu_n\big(M_\eta(x)\big) \leq \mu_n\big(\liminf_k M_\eta(x_k)\big) \leq \liminf_k \mu_n\big(M_\eta(x_k)\big)$$

Taking $\liminf_n$ on both sides, implies

$$\begin{aligned}\liminf_n \mu_n\big(M_\eta(x)\big) &\leq \liminf_n \liminf_k \mu_n\big(M_\eta(x_k)\big) \\ &\leq \liminf_n \mu_n\big(M_\eta(x_n)\big) \qquad \text{(diagonal sequence)}\end{aligned}$$

$M_\eta(x)$ open $\Longrightarrow$ Portmanteau Theorem implies:

$$\mu\big(M_\eta(x)\big) \leq \liminf_n \mu_n\big(M_\eta(x)\big)$$

Hence,

$$\mu\big(M_\eta(x)\big) \leq \liminf_n \mu_n\big(M_\eta(x_n)\big) \quad \forall \eta \in I\!R$$

and

$$\nu(a \leq \eta) + \mu\big(M_\eta(x)\big) \leq \nu(a \leq \eta) + \liminf_n \mu_n\big(M_\eta(x_n)\big) \leq 1 \quad \forall \eta \in I\!R$$

q.e.d.

Lemma 2 ("Portmanteau for Integrals"):

$\mathsf{f} \geq 0$ and lsc, $\quad \mu_n \xrightarrow{w} \mu \quad \Longrightarrow \quad \int \mathsf{f}\, d\mu \;\leq\; \liminf_n \int \mathsf{f}\, d\mu_n$

Sketch of Proof:

For bounded f, i.e., $0 < \mathsf{f} < 1$. Full result by truncation and mon. conv.

Consider for fixed k: $\quad F_i := \{\tau \,:\, i/k < \mathsf{f}(\tau)\}$, which is open for all i.

$$\Rightarrow \;\; \tfrac{1}{k} \textstyle\sum_{i=1}^{k} \mu(F_i) \;\leq\; \int \mathsf{f}\, d\mu \;\leq\; \tfrac{1}{k} + \tfrac{1}{k} \sum_{i=1}^{k} \mu(F_i)$$

Portmanteau $\;\Rightarrow\;$ $\mu(F_i) \;\leq\; \liminf_n \mu_n(F_i) \quad \forall i$

$$\Rightarrow \;\; -\frac{1}{k} + \int \mathsf{f}\, d\mu \;\leq\; \frac{1}{k} \sum_{i=1}^{k} \mu(F_i) \;\leq\; \frac{1}{k} \liminf_n \sum_{i=1}^{k} \mu(F_i) \;\leq\; \liminf_n \int \mathsf{f}\, d\mu_n$$

q.e.d.

Proposition 2 (for $i = 2$):

$$(A1) + (A2) \quad \Longrightarrow \quad C : \mathcal{P}_1(I\!R^s) \longrightarrow 2^{I\!R^m} \text{ closed.}$$

$$\mathcal{P}_1(I\!R^s) \;:=\; \{\mu \in \mathcal{P}_1(I\!R^s) \,:\, \int \|z\|\, \mu(dz) \;<\; +\infty\}$$

Proof:

$$x_n \in C(\mu_n) \quad \text{iff} \quad \int_{I\!R^s} [f(x_n, z) - \eta]_+ \, \mu_n(dz) \;\leq\; a_\eta \qquad \forall \eta \in I\!R$$

$[f(\cdot,\cdot) - \eta]_+ \geq 0$, lsc, and Fatou-Lemma imply

$$\begin{aligned} \int [f(x,z) - \eta]_+ \, \mu_n(dz) \quad &\leq \quad \int \liminf_k \, [f(x_k, z) - \eta]_+ \, \mu_n(dz) \\ &\leq \quad \liminf_k \int [f(x_k, z) - \eta]_+ \, \mu_n(dz) \end{aligned}$$

taking $\liminf_n$ on both sides implies

$$\begin{aligned}
\liminf_n \int [f(x,z)-\eta]_+ \, \mu_n(dz) &\leq \liminf_n \, \liminf_k \int [f(x_k,z)-\eta]_+ \, \mu_n(dz) \\
&\leq \liminf_n \int [f(x_n,z)-\eta]_+ \, \mu_n(dz) \\
&\leq a_\eta \qquad \forall \eta \in \mathbb{R}
\end{aligned}$$

“Portmanteau for Integrals” implies

$$\int [f(x,z)-\eta]_+ \, \mu(dz) \;\leq\; \liminf_n \int [f(x_n,z)-\eta]_+ \, \mu_n(dz) \;\leq\; a_\eta \qquad \forall \eta \in \mathbb{R}$$

q.e.d.

Comments and Corollaries:

- $C(\mu)$ is a closed set for any $\mu \in \mathcal{P}(I\!R^s)$ $(i = 1)$ and any $\mu \in \mathcal{P}_1(I\!R^s)$ $(i = 2)$.

- Stability Analysis:
  If g is lsc and X non-empty, compact then $\varphi : \mathcal{P}_{(1)}(I\!R^s) \longrightarrow I\!R$ with

$$\varphi(\mu) \ := \ \min\{g(x) \ : \ f(x,z) \succeq_i a, \, x \in X\}, \ \ i = 1, 2$$

  is lsc.

- Related Work:
  Dentcheva/Henrion/Ruszczyński (2005) have studied stability of dominance constraints with abstract random variables fulfilling stronger continuity properties. Both μ and ν are subjected to peturbations under suitable discrepancies.

Algorithmic Aspects

Reductions for discrete random variables

z, a discrete with realizations $z_l, l = 1, \ldots, L$, and $a_k, k = 1, \ldots, K$.

First-Order Stochastic Dominance

$$\nu(a \leq \eta) \quad \leq \quad \mu\big(f(x,z) \leq \eta\big) \qquad \forall \eta \in I\!R$$

if and only if

$$\nu(a \leq a_k) \quad \leq \quad \mu\big(f(x,z) \leq a_k\big) \qquad k = 1, \ldots, K$$

(piece-wise constant, non-decreasing functions with values in $[0,1]$)

Second-Order Stochastic Dominance

$$\int_{I\!R^s} [f(x,z) - \eta]_+ \, \mu(dz) \;\le\; \int_{I\!R} [a - \eta]_+ \, \nu(da) \qquad \forall \eta \in I\!R$$

if and only if

$$\int_{I\!R^s} [f(x,z) - a_k]_+ \, \mu(dz) \;\le\; \int_{I\!R} [a - a_k]_+ \, \nu(da) \qquad k = 1, \ldots, K$$

(piece-wise linear, non-increasing convex functions with gradients in $[-1, 0]$)

Equivalent MILP – First-Order Stochastic Dominance

$$\min\Big\{g^\top x \ : \ x \in X,\ \nu(a \le a_k) \ \le \ \mu\big(f(x,z) \le a_k\big) \ \ \forall k\Big\}$$

$$= \min\Big\{g^\top x \ : \ x \in X,\ \mu\big(f(x,z) > a_k\big) \ \le \ 1 - \nu(a \le a_k) \ \ \forall k\Big\}$$

$f(x,z) = c^\top x + \min\{q^\top y \ : \ Wy = z - Tx,\ y \in Y\},\ \alpha_k := 1 - \nu(a \le a_k)$

$$\min\Big\{g^\top x \ : \ \begin{array}{rcll} c^\top x + q^\top y_{lk} - a_k & \le & M\theta_{lk} & \forall l\ \forall k \\ Tx + Wy_{lk} & = & z_l & \forall l\ \forall k \\ \sum_{l=1}^{L} \pi_l \theta_{lk} & \le & \alpha_k & \forall k \\ & & x \in X,\ y_{lk} \in Y,\ \theta_{lk} \in \{0,1\} & \forall l\ \forall k \end{array}\Big\}$$

Equivalent MILP – Second-Order Stochastic Dominance

$$\min \left\{ g^\top x \ : \ x \in X, \ \int_{I\!R^s} [f(x,z) - a_k]_+ \, \mu(dz) \ \leq \ \int_{I\!R} [a - a_k]_+ \, \nu(da) \ \ \forall k \right\}$$

$$\min \left\{ g^\top x \quad : \quad \begin{array}{rcll} c^\top x + q^\top y_{lk} - a_k & \leq & v_{lk} & \forall l \ \forall k \\ Tx + W y_{lk} & = & z_l & \forall l \ \forall k \\ \sum_{l=1}^{L} \pi_l v_{lk} & \leq & \bar{\alpha}_k & \forall k \end{array} \right.$$

$$\left. x \in X, \ y_{lk} \in Y, \ v_{lk} \geq 0 \ \ \forall l \ \forall k \right\}$$

$$\bar{\alpha}_k := \int_{I\!R} [a - a_k]_+ \, \nu(da)$$

Idea of proof: $\{l : f(x, z_l) - a_k > 0\} \subseteq \{l : v_{lk} > 0\}$

Remarks

- Additional coupling prevents direct application of scenario decomposition.
- Second-order problem is relaxation of first-order problem.
- Whithout integer components in X and Y:
  - First-order model allows cutting plane (decomposition) approach.
  - Second-order problem is convex (LP).

Algorithm: Decomposition by Lagrangean Relaxation

$$\min \Big\{ g^\top x \quad : \quad \begin{array}{rcll} c^\top x + q^\top y_{lk} - a_k & \leq & M\theta_{lk} & \forall l\ \forall k \\ Tx + W y_{lk} & = & z_l & \forall l\ \forall k \\ \sum_{l=1}^{L} \pi_l \theta_{lk} & \leq & \alpha_k & \forall k \end{array}$$
$$x \in X,\ y_{lk} \in Y,\ \theta_{lk} \in \{0,1\} \quad \forall l\ \forall k \Big\}$$

in addition: Relaxation of Nonanticipativity

Lagrangean function:

$$\begin{aligned}\mathcal{L}(x,\theta,\lambda) &= \sum_{l=1}^{L}\pi_l\cdot g^\top x_l + \sum_{l=1}^{L}\sum_{k=1}^{K}\lambda_k\cdot(\pi_l\theta_{lk}-\pi_l\alpha_k)\\ &= \sum_{l=1}^{L}\mathcal{L}_l(x_l,\theta_l,\lambda)\end{aligned}$$

where

$$\mathcal{L}_l(x_l,\theta_l,\lambda) := \pi_l\cdot g^\top x_l + \pi_l\sum_{k=1}^{K}\lambda_k\cdot(\theta_{lk}-\alpha_k)$$

Lagrangean minimization:

$$\begin{aligned} D(\lambda) \;=\; \min \Big\{ \mathcal{L}(x,\theta,\lambda) \quad : \quad & c^\top x_l + q^\top y_{lk} - a_k \;\leq\; M\theta_{lk} \quad \forall l\;\forall k \\ & T x_l + W y_{lk} \;=\; z_l \quad \forall l\;\forall k \\ & x_l \in X,\; y_{lk} \in Y,\; \theta_{lk} \in \{0,1\} \quad \forall l\;\forall k \Big\} \end{aligned}$$

$$\begin{aligned} = \sum_{l=1}^{L} \min \Big\{ \mathcal{L}_l(x_l,\theta_l,\lambda) \quad : \quad & c^\top x_l + q^\top y_{lk} - a_k \;\leq\; M\theta_{lk} \quad \forall k \\ & T x_l + W y_{lk} \;=\; z_l \quad \forall k \\ & x_l \in X,\; y_{lk} \in Y,\; \theta_{lk} \in \{0,1\} \quad \forall k \Big\} \end{aligned}$$

Decomposition into single-scenario subproblems !

Lagrangean Dual:

$$\max \left\{ D(\lambda) \quad : \quad \lambda \in \mathbb{R}^K \right\}$$

- non-smooth convex minimization problem in tractable dimension,
- optimal value provides lower bound,
- feasibility heuristics provide upper bound,
- embedding into branch-and-bound in spirit of global optimization (subdivision of X).

Computations

Dispersed-generation instances

Sizes of first-order dominance models ($K = 4$):

Number of	10 scenarios	20 scenarios	30 scenarios	50 scenarios
Boolean variables	299.159	596.799	894.439	1.489.719
continuous variables	283.013	564.613	846.213	1.409.413
constraints	742.648	1.481.568	2.220.488	3.698.328

Scen.	Inst.	Benchmarks		Time (sec.)	Cplex		ddsip.vSD	
		Prob.	Value		Upper Bound	Lower Bound	Upper Bound	Lower Bound
10	1	0.12	2895000	430.43	–	29	29	15
		0.21	4851000	899.16	–	29	29	29
		0.52	7789000	15325.75	29	29	29	29
		0.15	10728000					
	2	0.12	2900000	192.48	–	27	28	15
		0.21	4860000	418.90	28	28	28	15
		0.52	7800000	802.94	28	28	28	28
		0.15	10740000					
	3	0.12	3000000	144.63	–	21	21	12
		0.21	5000000	428.61	21	21	21	18
		0.52	8000000	678.79	21	21	21	21
		0.15	11000000					
	4	0.12	3500000	164.34	–	11	13	10
		0.21	5500000	818.26	–	12	13	13
		0.52	8500000	28800.00	13	12	13	13
		0.15	11500000					
	5	0.12	4000000	171.52	–	7	8	8
		0.21	6000000	3304.02	8	8	8	8
		0.52	9000000					
		0.15	12000000					

Scen.	Inst.	Benchmarks		Time (sec.)	Cplex		ddsip.vSD	
		Prob.	Value		Upper Bound	Lower Bound	Upper Bound	Lower Bound
20	1	0.105	2895000	306.89	–	29	29	12
		0.1	4851000	1151.95	–	29	29	29
		0.69	7789000	9484.97	29	29	29	29
		0.105	10728000					
	2	0.105	2900000	703.91	–	27	28	18
		0.1	4860000	1744.75	28	28	28	26
		0.69	7800000	1916.06	28	28	28	28
		0.105	10740000					
	3	0.105	3000000	305.84	–	21	21	10
		0.1	5000000	1682.93	21	21	21	19
		0.69	8000000	2138.94	21	21	21	21
		0.105	11000000					
	4	0.105	3500000	425.98	–	11	13	9
		0.1	5500000	2213.08	–	12	13	13
		0.69	8500000	11236.31	–	12 m.	13	13
		0.105	11500000					
	5	0.105	4000000	447.33	–	8	8	8
		0.1	6000000	5599.99	9	8	8	8
		0.69	9000000	7840.09	9	8 m.	8	8
		0.105	12000000					

Scen.	Inst.	Benchmarks		Time (sec.)	Cplex		ddsip.vSD	
		Prob.	Value		Upper Bound	Lower Bound	Upper Bound	Lower Bound
30	1	0.085	2895000	473.27	–	28	29	12
		0.14	4851000	1658.02	–	29	29	29
		0.635	7789000	3255.99	–	29 m.	29	29
		0.14	10728000					
	2	0.085	2900000	1001.53	–	26	28	18
		0.14	4860000	2694.93	–	27	28	28
		0.635	7800000	3372.24	–	27 m.	28	28
		0.14	10740000					
	3	0.085	3000000	469.93	–	17	23	10
		0.14	5000000	3681.15	–	18 m.	21	20
		0.635	8000000	28800.00	–	–	21	20
		0.14	11000000					
	4	0.085	3500000	618.21	–	10	14	8
		0.14	5500000	3095.02	–	11 m.	14	10
		0.635	8500000	28800.00	–	–	14	13
		0.14	11500000					
	5	0.085	4000000	672.73	–	7	8	8
		0.14	6000000	8504.88	–	8 m.	8	8
		0.635	9000000					
		0.14	12000000					

Scen.	Inst.	Benchmarks		Time (sec.)	Cplex		ddsip.vSD	
		Prob.	Value		Upper Bound	Lower Bound	Upper Bound	Lower Bound
50	1	0.09	2895000	745.87	–	–	29	11
		0.135	4851000	2534.21	–	–	29	29
		0.67	7789000					
		0.105	10728000					
	2	0.09	2900000	1549.22	–	–	28	18
		0.135	4860000	4168.89	–	–	28	28
		0.67	7800000					
		0.105	10740000					
	3	0.09	3000000	756.06	–	–	23	10
		0.135	5000000	28800.00	–	–	21	20
		0.67	8000000					
		0.105	11000000					
	4	0.09	3500000	975.20	–	–	15	8
		0.135	5500000	28800.00	–	–	13	12
		0.67	8500000					
		0.105	11500000					
	5	0.09	4000000	1150.95	–	–	8	8
		0.135	6000000					
		0.67	9000000					
		0.105	12000000					

Computations with dispersed-generation instances

Second-order dominance models

20 data and 4 benchmark scenarios:

Inst.	Benchmarks		Time (sec.)	Cplex		ddsip.vSD	
	Prob.	Value		Upper Bound	Lower Bound	Upper Bound	Lower Bound
1	0.105	2895000	455.36	–	29	29	9
	0.1	4851000	1602.92	–	29	29	29
	0.69	7789000	3694.92	29	29	29	29
	0.105	10728000					
2	0.105	2900000	464.26	–	27	27	9
	0.1	4860000	2501.01	–	27	27	27
	0.69	7800000	4661.86	27	27	27	27
	0.105	10740000					
3	0.105	3000000	362.91	–	18	18	9
	0.1	5000000	3101.28	–	18	18	18
	0.69	8000000	3952.93	18	18	18	18
	0.105	11000000					
4	0.105	3500000	386.32	–	11	11	9
	0.1	5500000	1197.34	–	11	11	11
	0.69	8500000	3760.04	11	11	11	11
	0.105	11500000					
5	0.105	4000000	430.61	–	8	8	8
	0.1	6000000	3035.52	8	8	8	8
	0.69	9000000					
	0.105	12000000					

30 data and 4 benchmark scenarios:

Inst.	Benchmarks		Time (sec.)	Cplex		ddsip.vSD	
	Prob.	Value		Upper Bound	Lower Bound	Upper Bound	Lower Bound
1	0.085	2895000	697.21	–	29	29	9
	0.14	4851000	2471.63	–	29	29	29
	0.635	7789000	7520.07	29	29	29	29
	0.14	10728000					
2	0.085	2900000	702.31	–	27	27	9
	0.14	4860000	3635.25	–	27	27	27
	0.635	7800000	14905.68	–	27m.	27	27
	0.14	10740000					
3	0.085	3000000	666.31	–	18	18	9
	0.14	5000000	3907.92	–	18	18	18
	0.635	8000000	7181.68	18	18	18	18
	0.14	11000000					
4	0.085	3500000	500.05	–	11	11	9
	0.14	5500000	1404.96	–	11	11	11
	0.635	8500000	6559.52	11	11	11	11
	0.14	11500000					
5	0.085	4000000	474.68	–	8	8	8
	0.14	6000000	6076.40	8	8	8	8
	0.635	9000000					
	0.14	12000000					

50 data and 4 benchmark scenarios:

Inst.	Benchmarks		Time (sec.)	Cplex		ddsip.vSD	
	Prob.	Value		Upper Bound	Lower Bound	Upper Bound	Lower Bound
1	0.09	2895000	1084.68	–	–	29	9
	0.135	4851000	3747.69	–	–	29	29
	0.67	7789000					
	0.105	10728000					
2	0.09	2900000	1125.39	–	–	27	9
	0.135	4860000	5857.67	–	–	27	27
	0.67	7800000					
	0.105	10740000					
3	0.09	3000000	1041.15	–	–	18	9
	0.135	5000000	6126.89	–	–	18	18
	0.67	8000000					
	0.105	11000000					
4	0.09	3500000	1026.21	–	–	11	9
	0.135	5500000	2872.83	–	–	11	11
	0.67	8500000					
	0.105	11500000					
5	0.09	4000000	1096.69	–	–	8	8
	0.135	6000000					
	0.67	9000000					
	0.105	12000000					

Publications:

Gollmer, R.; Neise, F.; Schultz, R.:

Stochastic Programs with First-Order Dominance Constraints Induced by Mixed-Integer Linear Recourse

Gollmer, R.; Gotzes, U.; Schultz, R.:

Second-Order Stochastic Dominance Constraints Induced by Mixed-Integer Linear Recourse