A Risk-Averse Newsvendor Problem
With Coherent Measures of risk

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Purpose and Overview of the study

- Apply the risk theory to analyze a risk-averse newsvendor problem
- Review the effects of the degree of risk-aversion to the optimal solution
- Extend the results to all law-invariant coherent measures of risk
- Confirm the implications by numerical results

Content of the study

- Mean-risk model, Coherent measure of risk and Law-invariant coherent measure of risk
- Model description
- Risk-neutral and Risk-averse newsvendor solutions
- Extension to general law-invariant coherent measure of risk
- Numerical examples

Mean-Risk Model

- Minimize $\rho(X) = -\mathbf{E}[X] + \lambda \mathbf{r}[X]$

Coherent Risk Measure

- Convexity: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)\ \forall\ X, Y \in \aleph$ and $\alpha \in [0, 1]$.

- Monotonicity: If $X, Y \in \aleph$ and $Y \succ X$, then $\rho(Y) \geq \rho(X)$.

- Translation Equivalence: If $a \in \mathbf{R}$ and $X \in \aleph$, then $\rho(X + a) = \rho(X) - a$.

- Positive Homogeneity: If $t > 0$ and $X \in \aleph$, then $\rho(tX) = t \cdot \rho(X)$.

Law-invariant coherent measure of risk

- Definition
  - $\rho[Z_1] = \rho[Z_2]$ whenever Z_1 and Z_2 have the same distribution

Examples of $r[\cdot]$ such that $-\mathbf{E}[X] + \lambda \mathbf{r}[X]$ is law-invariant coherent

- Absolute deviation from a quantile: $(\lambda \in [0, 1])$

$$
\begin{aligned}
\mathbf{r}_\alpha(X) &= \min_{\eta \in \mathbf{R}} \mathbf{E}[\max((1-\alpha)\cdot(\eta - X), \alpha(X-\eta))] \\
&= \mathbf{E}[\max((1-\alpha)\cdot(q_\alpha - X), \alpha(X - q_\alpha))] \text{ where } \eta^* = q_\alpha(X)
\end{aligned}
$$

- Semideviation: Expectation of the shortfall $(\lambda \in [0, 1])$

$$
\begin{aligned}
\sigma_p(X) &= (\mathbf{E}[(\mathbf{EX} - X)_+^p])^{1/p} : p \geq 1 \\
\text{if } p = 1,\ \sigma_1(X) &= \max_{0<\beta<1} \mathbf{r}_\beta[X]
\end{aligned}
$$

Model Description

- Parameters and Decision variables
  - p: unit resale price per item
  - c: unit ordering cost per item
  - s: unit salvage value and we assume that $p > c > s$
  - D: demand (random variable) with mean μ
  - x: amount of order (decision variable)
  - $Z(x)$: net profit on $x = -c \cdot x + p \cdot \min(D, x) + s \cdot (x - D)_+$
    $= -\bar{c} \cdot x + \bar{p} \cdot \min(x, D)$ where $\bar{p} = p - s > 0$ and $\bar{c} = c - s > 0$

- Objective function

$$\max_x \; -\rho(Z(x)) = \mathbf{E}(Z(x)) - \lambda \cdot \mathbf{r}(Z(x)), \qquad 0 \leq \lambda \leq 1$$

Risk-neutral Newsvendor Solution

- $E[Z(x+1,D)] - E[Z(x,D)] = 0$
  $\Rightarrow \hat{x}_{RN} \in [q_\alpha^-(D), q_\alpha^+(D)]$: well-known solution
  where $\alpha = \frac{(\bar{p}-\bar{c})}{\bar{p}}$ and $q_\alpha^-(D), q_\alpha^+(D)$: left and right α-quantile of D.

$$\begin{aligned} q_\alpha^-(D) &= \inf\left\{\eta : \mathbf{P}[D \le \eta] \ge \alpha\right\} \\ q_\alpha^+(D) &= \sup\left\{\eta : \mathbf{P}[D < \eta] \le \alpha\right\} \end{aligned}$$

Risk-averse Newsvendor Solution

- Lemma 1. $\sigma_1[Z(x)]$ and $\mathbf{r}_\beta[Z(x)]$ are nondecreasing functions for x

  The more we order, the larger $\mathbf{r}_\beta[Z(x)]$ is, which is more risky situation.

- Theorem 1. $\hat{x}_{RA} \leq \hat{x}_{RN}$ with a risk functional $\mathbf{r}_\beta[Z(x)]$

  The more risk-averse the newsvendor is, the less the order is.

Extension to Law-invariant coherent measure of risk

- Kusuoka Thereom(Kusuoka, 2001): For every lower semicontinuous law-invariant coherent measure of risk, $\rho[\cdot]$ on $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathcal{P})$,
  $\exists$ a convex set $\mathcal{M}$ of probability measures on $[0, 1]$ such that

$$\begin{aligned}\rho[Z] &= -\mathbf{E}[Z] + \varkappa_{\mathcal{M}}[Z] \\ &= -\mathbf{E}[Z] + \sup_{\mu \in \mathcal{M}} \int_0^1 \frac{1}{\beta} r_\beta[Z] \, \mu(d\beta)\end{aligned}$$

- Theorem 2. The results from lemma (1) and theorem (2) are still true with law-invariant coherent measure of risk
  - The mean-risk model with all law-invariant coherent measures of risk can be derived from the model with $\mathbf{r}_\beta[Z(x)]$

Practical importance of law-invariant coherent measure of risk

- We derived the monotonicity of $\mathbf{r}[\cdot]$ on x and of $\widehat{x}_{RA}$ with respect to λ with $\mathbf{r}_\beta[Z(x)]$. That is, we only proved our conjectures with only one risk functional, $\mathbf{r}_\beta[Z(x)]$

- With setting proper convex set $\mathcal{M}$ of probability measure, we can generalize our results to all law-invariant coherent measure of risk and constructively define the risk measure

Sample-based optimization

- LP formulation for semideviation

$$\begin{array}{rl} \text{max} & \rho = \sum_{k=1}^{N} p_k Z_k - \lambda \sum_{k=1}^{N} p_k \mathbf{r_k} \\ \text{subject to} & Z_k \leq -\bar{c}x + \bar{p}D_k, \quad k = 1, \dots, N \\ & Z_k \leq -\bar{c}x + \bar{p}x, \quad k = 1, \dots, N \\ & \mathbf{r}_k \geq \mu - Z_k, \quad k = 1, \dots, N \\ & x \geq 0 \quad \text{and} \quad \mathbf{r}_k \geq 0, \quad k = 1, \dots, N. \end{array}$$

- LP formulation for absolute deviation from a quantile

$$\begin{array}{rl} \text{max} & \rho = \sum_{k=1}^{N} p_k Z_k - \lambda \sum_{k=1}^{N} p_k \cdot [(1-\alpha) \cdot w_k + \alpha \cdot v_k] \\ \text{subject to} & Z_k = \eta + v_k - w_k, \quad k = 1, \dots, N \\ & Z_k \leq -\bar{c}x + \bar{p}D_k, \quad k = 1, \dots, N \\ & Z_k \leq -\bar{c}x + \bar{p}x, \quad k = 1, \dots, N \\ & x \geq 0 \quad \text{and} \quad w_k \geq 0, v_k \geq 0 \quad k = 1, \dots, N. \end{array}$$

λ	Semideviation			Deviation from Median			Worst Scaled Deviation		
	$\hat{x}$	$E(Z)(\lambda)$	$r(Z)(\lambda)$	$\hat{x}$	$E(Z)(\lambda)$	$r(Z)(\lambda)$	$\hat{x}$	$E(Z)(\lambda)$	$r(Z)(\lambda)$
0	63.22	157.70	74.41	63.22	157.70	76.46	63.04	157.70	346.63
0.2	58.66	157.15	68.40	58.93	157.22	76.13	53.68	154.56	298.38
0.4	55.55	155.78	64.08	56.46	156.21	73.21	38.62	134.24	232.89
0.6	52.09	153.27	59.07	53.14	154.14	68.96	6.10	28.84	29.91
0.8	48.38	149.45	53.63	48.24	149.29	62.05	3.37	16.33	10.29
1.0	44.25	143.95	47.50	42.09	140.56	52.39	2.73	13.29	6.82

Table 1. Solutions for different levels of risk aversion for the uniform distribution

λ	Semideviation			Deviation from Median			Worst Scaled Deviation		
	$\hat{x}$	$E(Z)(\lambda)$	$r(Z)(\lambda)$	$\hat{x}$	$E(Z)(\lambda)$	$r(Z)(\lambda)$	$\hat{x}$	$E(Z)(\lambda)$	$r(Z)(\lambda)$
0	63.22	157.70	74.41	63.22	157.70	76.46	63.04	157.70	346.63
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1.0	44.25	143.95	47.50	42.09	140.56	52.39	2.73	13.29	6.82

Table 2. Solutions for different levels of risk aversion for the lognormal distribution

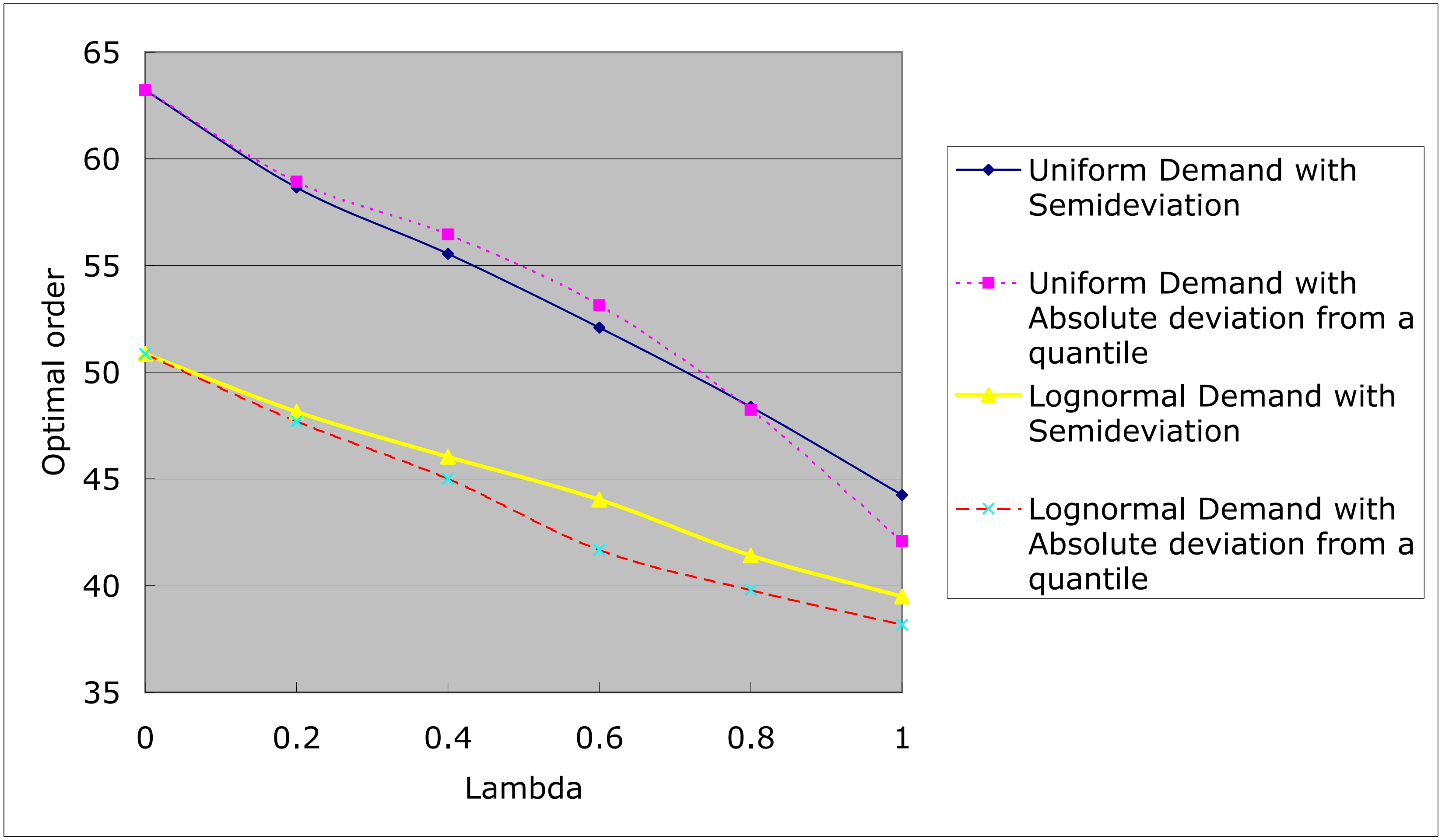

Figure 1. The optimal orders for different λ in each probability distribution

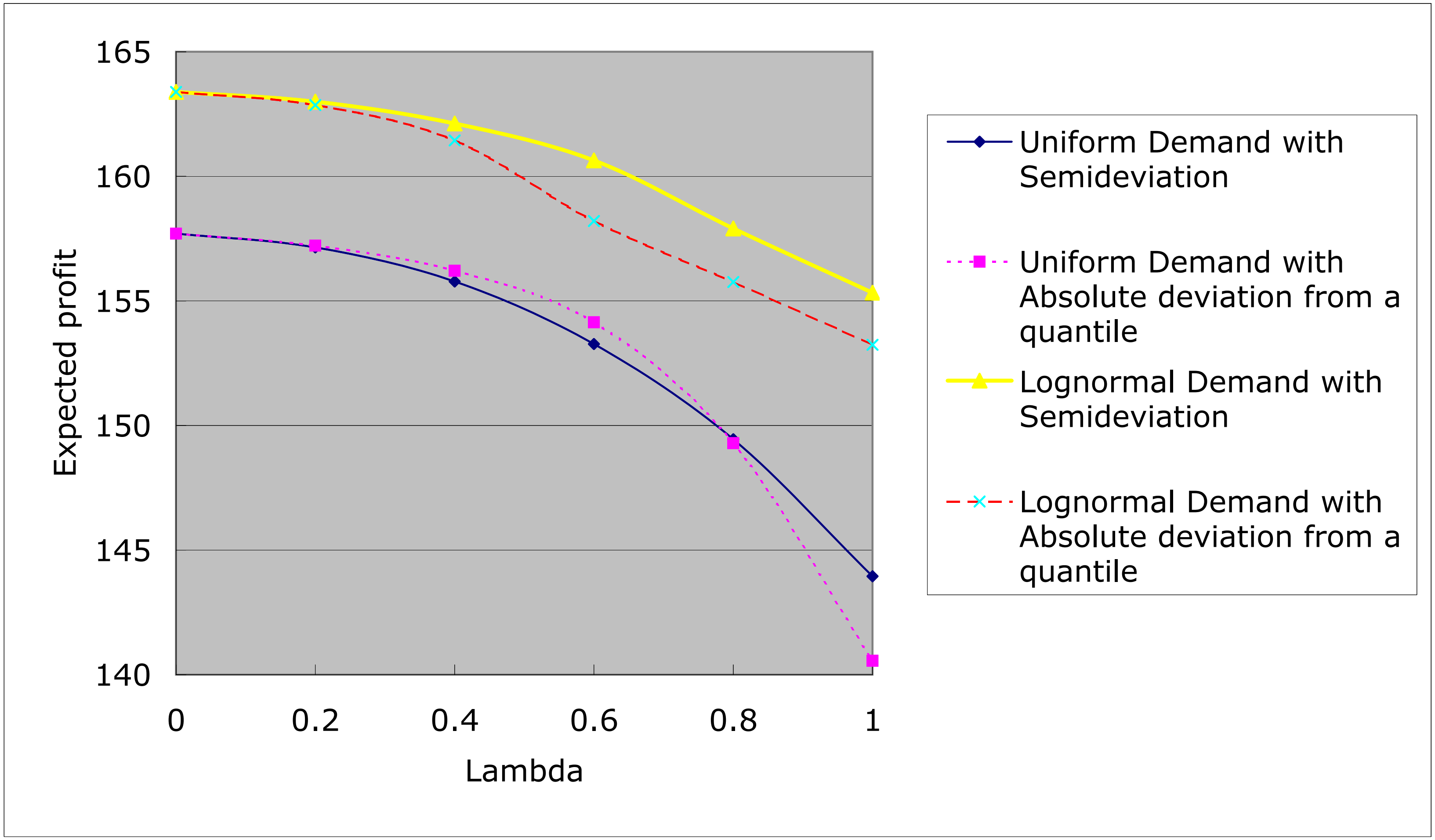

Figure 2. The expected profits at the optimal risk-averse solution for different λ in each probability distribution

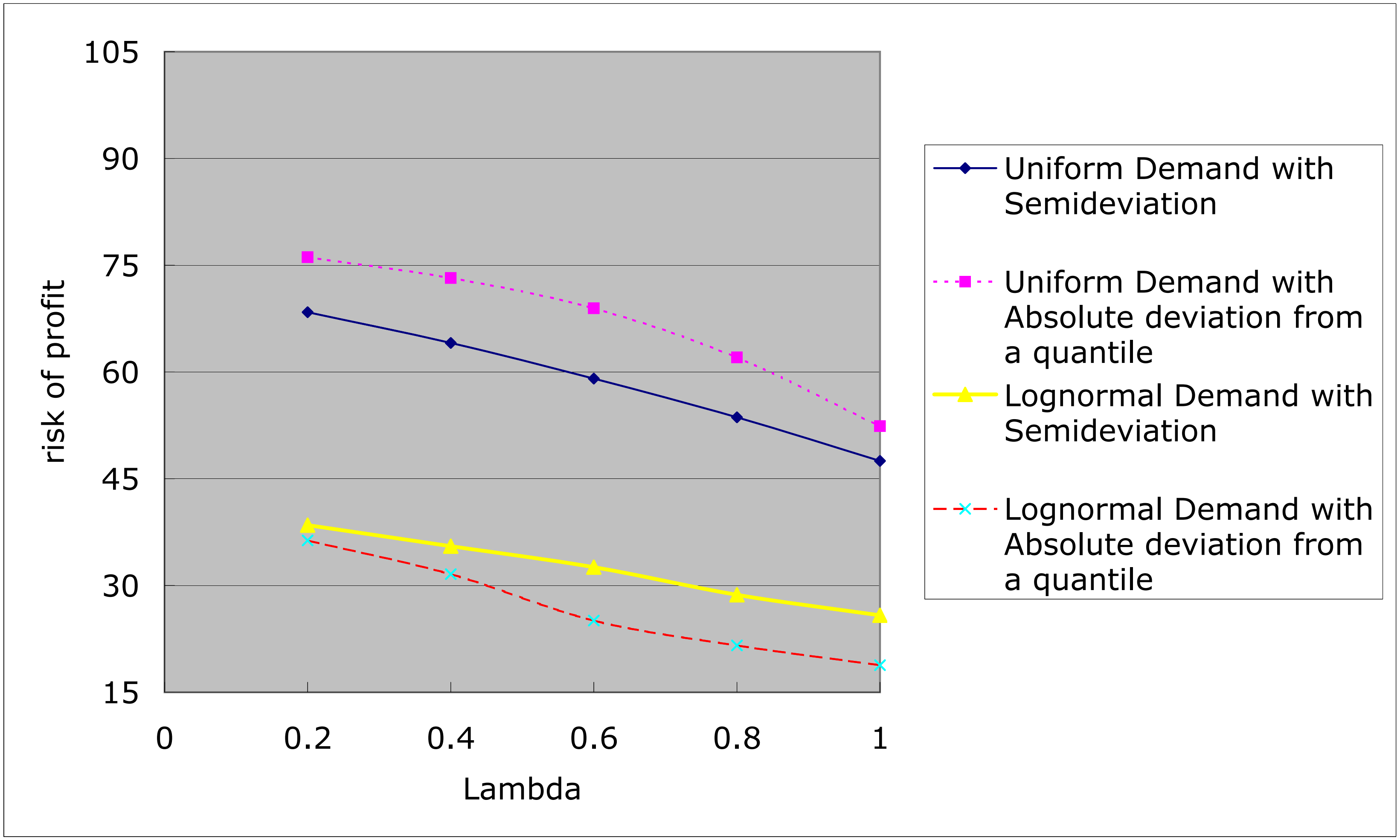

Figure 3. The risk of profit at the optimal risk-averse solution for different λ in each probability distribution

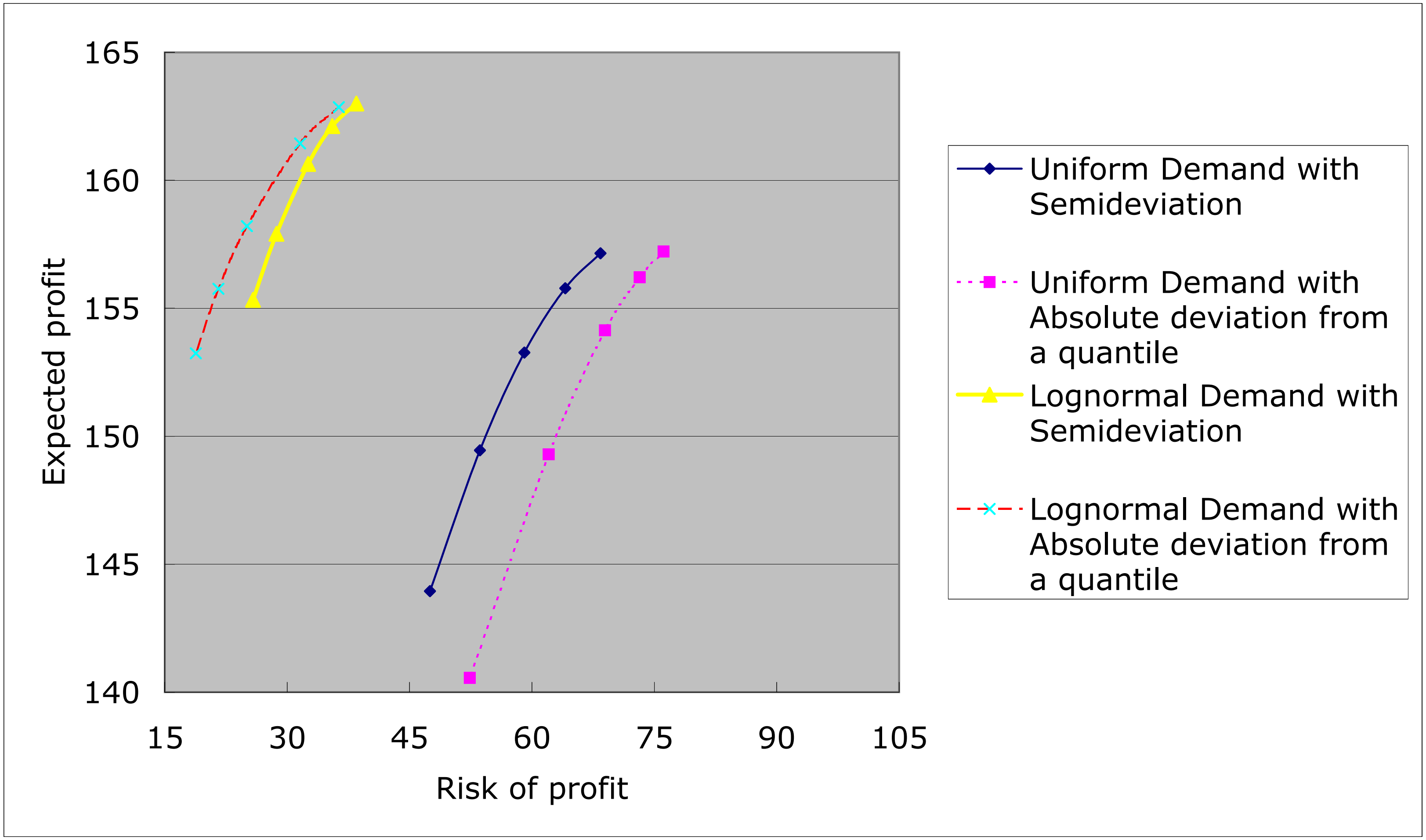

Figure 4. Efficient Frontiers for different λ in each probability distribution

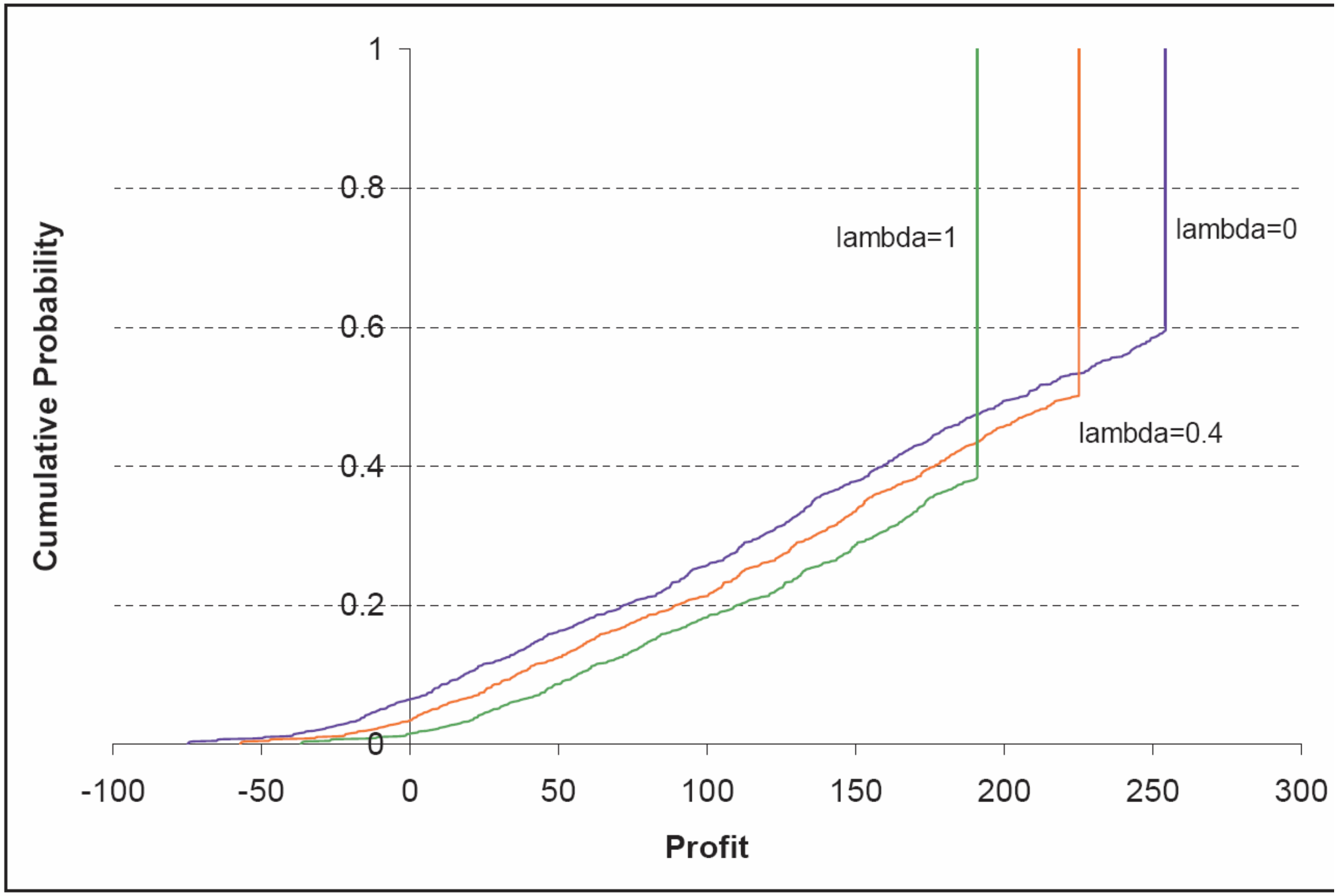

Fig 5: The cumulative distribution function of the profit for the profit for different levels of risk aversion in the problem with lognormal distribution. The mean deviation from median is used as the risk functional.

Further Research

- Multiproduct/Multistage newsvendor problem

- Other inventory models with risk measures