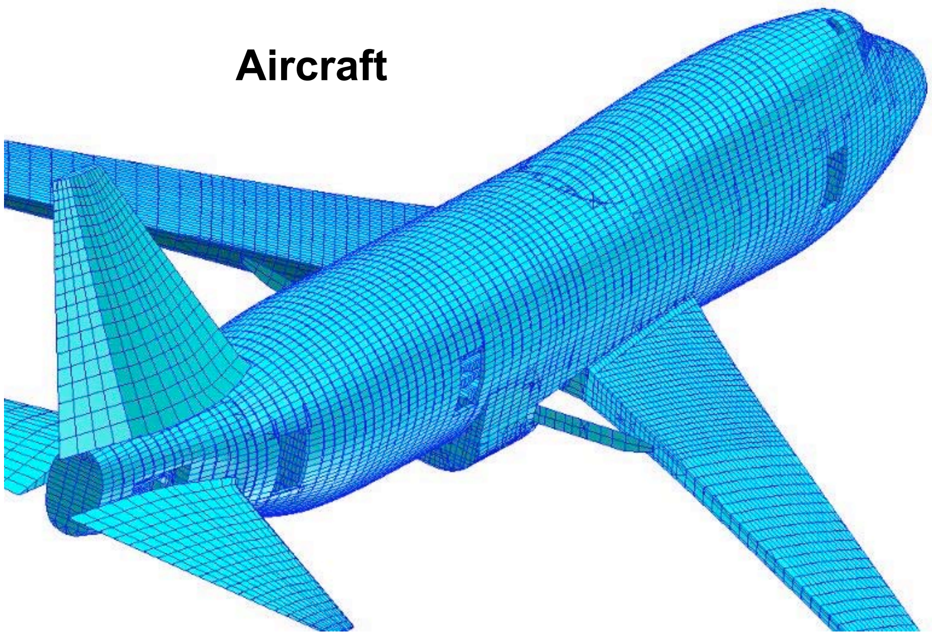
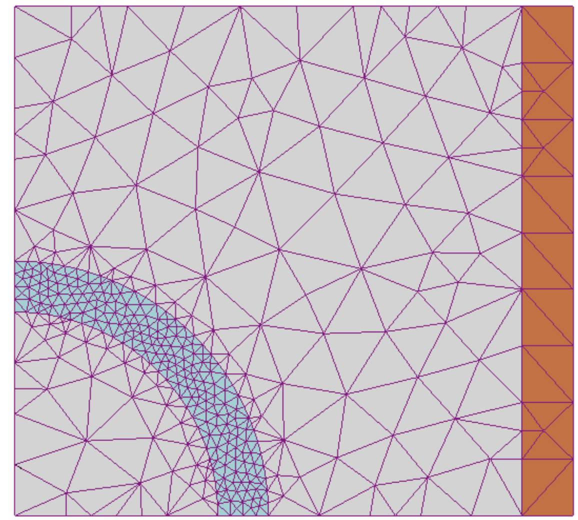
ME 345: Modeling & Simulation

Introduction to Finite Element Method

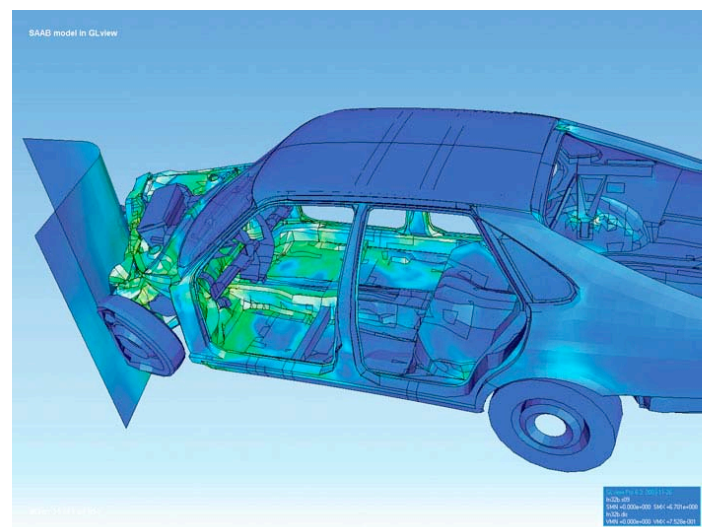
Examples

Aircraft



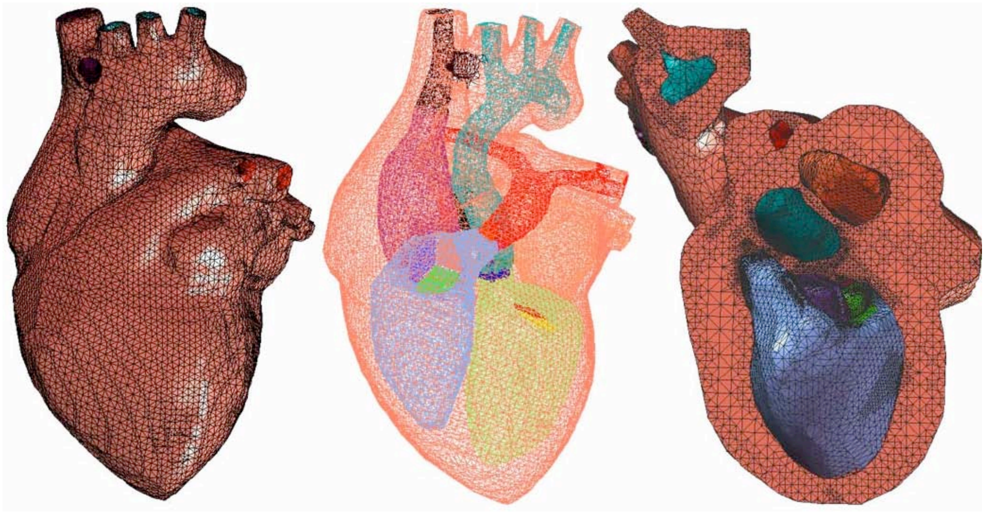


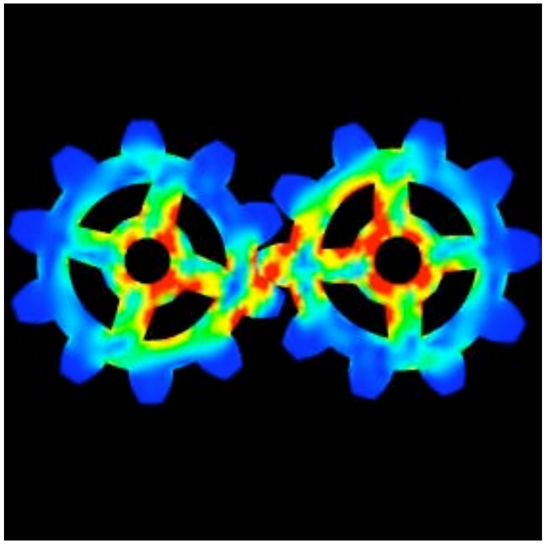
2D plate



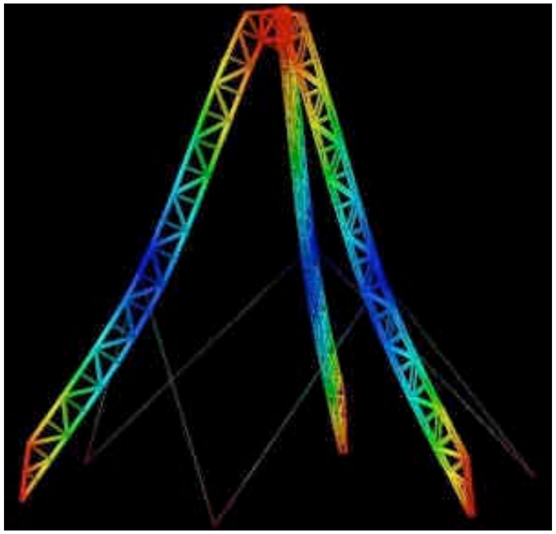
Crashworthiness

Human Heart

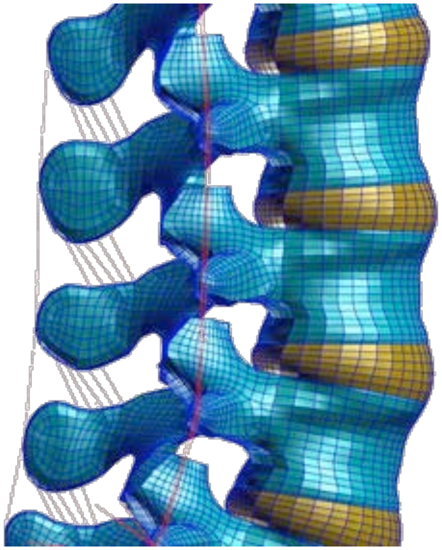




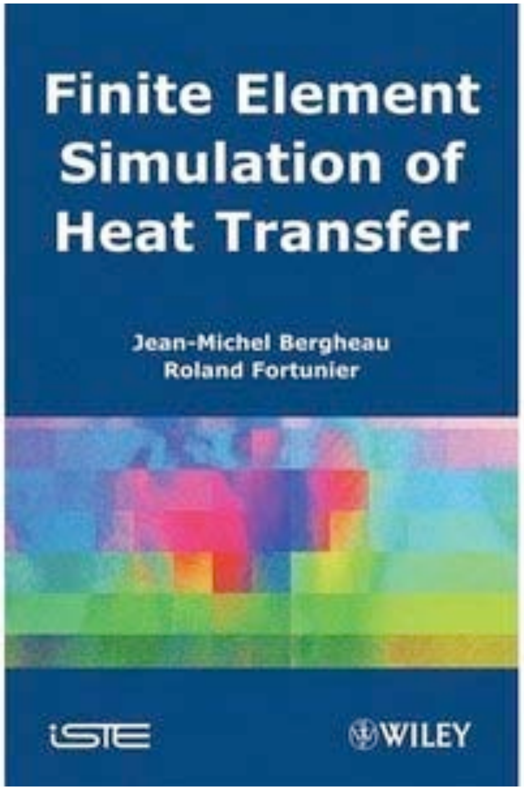
Gears

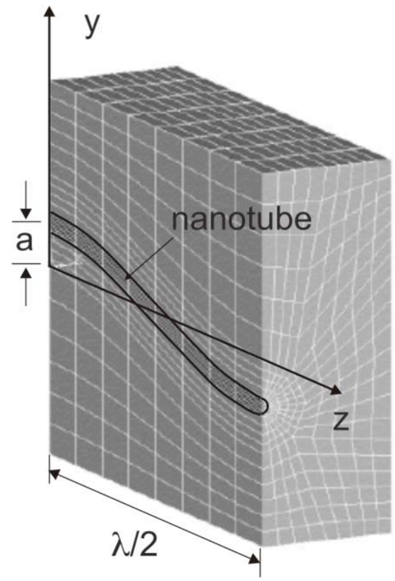


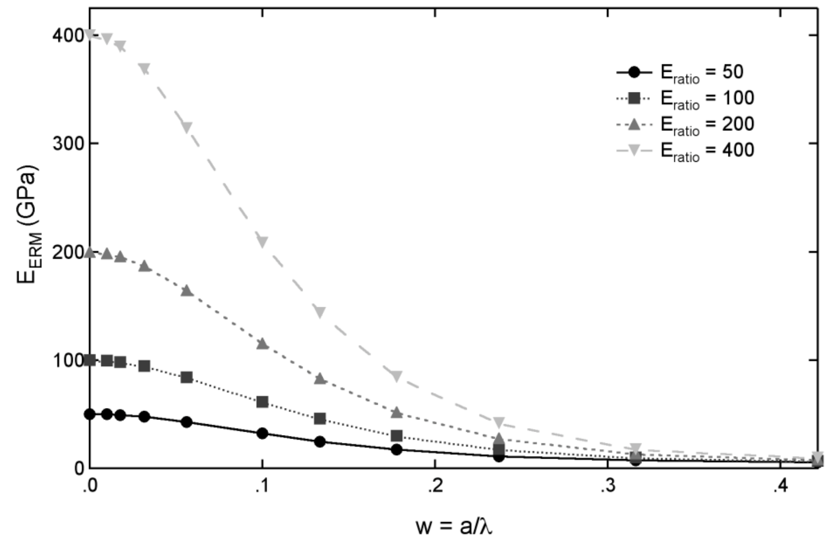
Structure



Human Spine

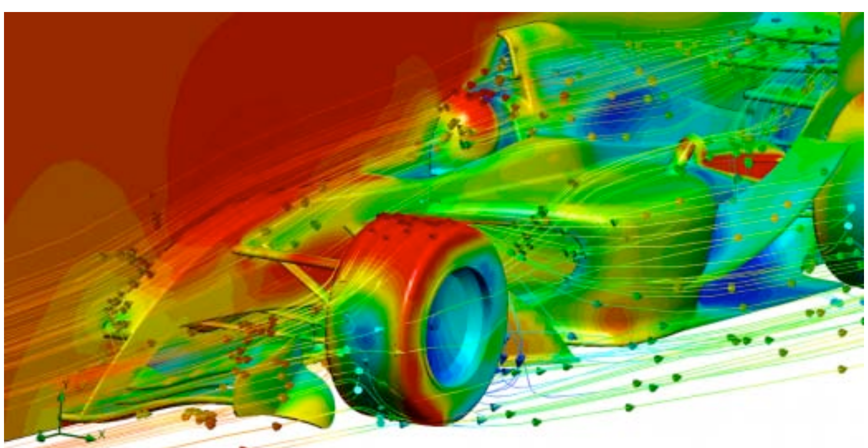




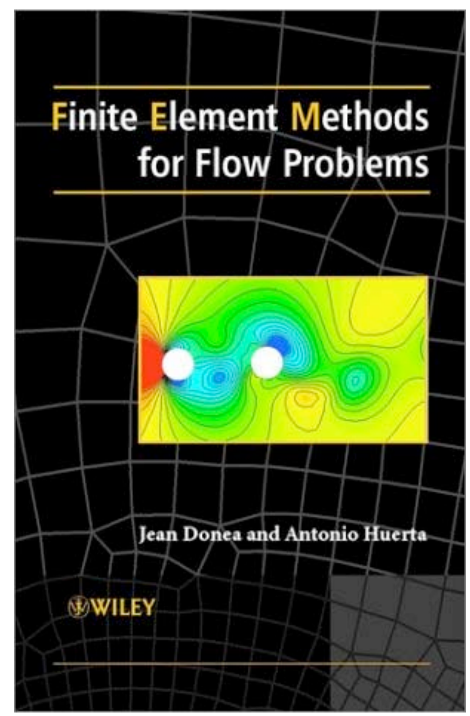


F.T. Fisher, PhD Dissertation, 2002

Fluid Flow over F1 racecar



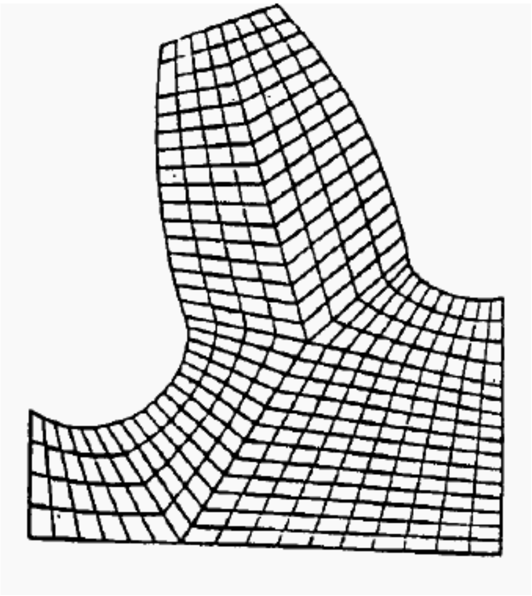
http://blogs.mentor.com/roi/blog/2009/11/17/optimizing-flow-fields-with-cfd/



Why use the finite element method? (FEM, FEA, FE, etc)

- One of the most commonly used methods of stress analysis
- Versatile computational tool
- Discretize (i.e. **approximate**) complex problems for which analytical solutions are difficult/impossible
- Increasingly easy to obtain results, but… are the results meaningful? (GIGO)
- SolidWorks/CosmosWorks, ProE, others… solid models directly into FEM analyzes
- More complex finite element packages available

Discretization (Meshing)



- Discretize a region into a *finite number of elements* (hence, FEM)
  - *Nodes:*
    - *simple definition: where elements meet*
    - *More complex definition:*
- Solve appropriate systems of equations given appropriate constraints
- **Assume linear elastic, static analysis**
  - Small displacements (negligible chance in geometry)
  - Response of the structure is static
  - Linear elastic material

Linear Static Stress Analysis

- Following conditions must be satisfied:

1. Equilibrium

$$\sum \mathbf{F} = \mathbf{0}$$

$$\sum \mathbf{M} = \mathbf{0}$$

2. Linear stress-strain law

$f = kx$	Linear elastic spring
$\sigma = E\varepsilon$	1D stress-strain law

3. Compatibility (strain-displacement conditions)
   - Continuous displacements
   - No gaps/overlaps in the body due to displacements

Element stiffness matrix: 1D spring

Assembly of global stiffness matrix

- Again, need to ensure: compatibility, equilibrium, and stress-strain relationships

- Note that the global stiffness matrix is BC independent!

Enforcement of boundary conditions

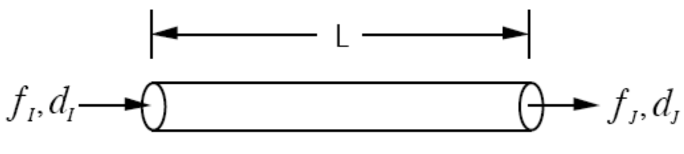
- The global stiffness matrix is *boundary condition independent*. The same “mesh” can be used to solve a class of problems of similar geometry.
- Boundary conditions: prescribed displacements and external forces at the nodes.
- Multiple methods of accomplishing this. Computationally, want the most efficient method.

Example problems: 1D spring

- Three problems using the same global stiffness matrix, but the matrix algebra is different after accounting for the BCs.
- Note: statically indeterminate “problems”… not a problem because additional geometry constraints are included in the finite element derivation.

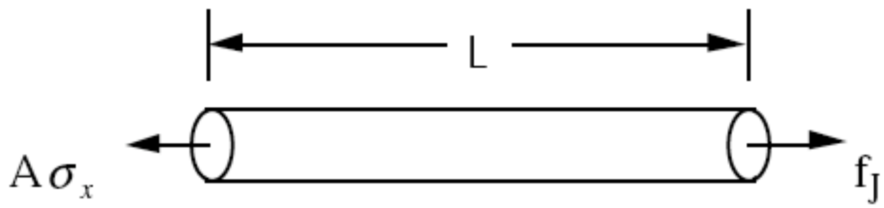
Element stiffness: 1D rod

A rod element is a one dimensional element similar to a spring.



$$\mathbf{f}_e = \begin{Bmatrix} f_I \\ f_J \end{Bmatrix} \quad \mathbf{d}_e = \begin{Bmatrix} d_I \\ d_J \end{Bmatrix}$$

Cross-sectional area A, Young's modulus E

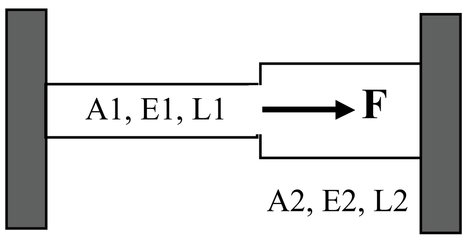


$$\begin{Bmatrix} f_I \\ f_J \end{Bmatrix} = \boxed{\frac{AE}{L} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}} \begin{Bmatrix} d_I \\ d_J \end{Bmatrix}$$

Element stiffness of a 1D rod (K_e)

Numerical example

- Once the element stiffness is determined, continue with the same steps as for the spring examples:
  - Assemble global stiffness matrix
  - Enforce boundary conditions
  - Matrix algebra



F = 1000 lbs
$A1 = 1\ in^2$, $A2 = 2\ in^2$
$E1 = 1 * 10^7$ psi, $E2 = 2 * 10^7$ psi
L1 = 10 in, L2 = 5 in

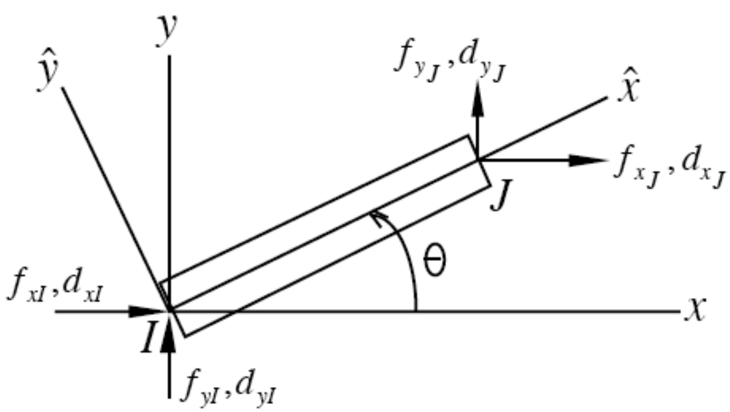
- Other variables can also be determined (strains, stresses, etc.)

Icicle Problem (simplified)

- Analytical solution, compare with FE code (Matlab)

Icicle Problem (more complex)

Stiffness method for 2D truss

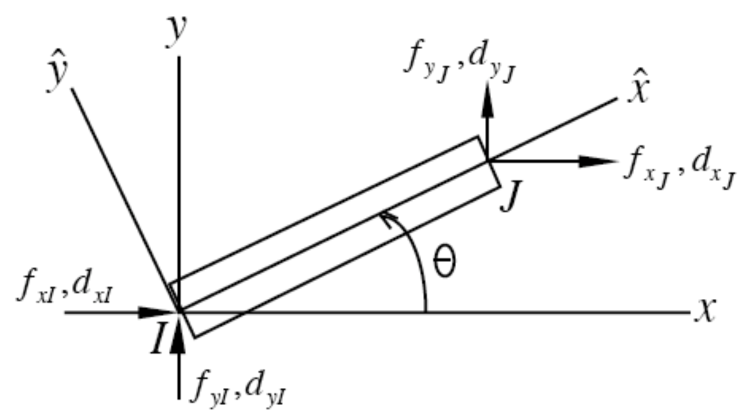


$$\begin{Bmatrix} \hat{f}_{xI} \\ \hat{f}_{yI} \\ \hat{f}_{xJ} \\ \hat{f}_{yJ} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{xI} \\ \hat{d}_{yI} \\ \hat{d}_{xJ} \\ \hat{d}_{yJ} \end{Bmatrix}$$

Element stiffness
in the element coordinate system.

Global Stiffness 2D truss elements

- Need to interconvert from global coordinates to local coordinates for each truss element
- For displacements, define transformation tensor *T*
- Similar transformation necessary for forces
- The conclusion is that…



$$\mathbf{K_e} = \mathbf{T}^{\mathrm{T}}\widehat{\mathbf{K}}_{\mathbf{e}}\mathbf{T}$$

$$= \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$\begin{Bmatrix} \hat{d}_{xI} \\ \hat{d}_{yI} \\ \hat{d}_{xJ} \\ \hat{d}_{yJ} \end{Bmatrix} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}}_{\Downarrow \; T}\begin{Bmatrix} d_{xI} \\ d_{yI} \\ d_{xJ} \\ d_{yJ} \end{Bmatrix}$$

Element stiffnessin the global coordinate system. (This is what we will want to use!)

Two dimensional example