ME345 – Modeling and Simulation
Professor Frank Fisher
Spring 2006
May 3, 2006

NAME: KEY

"I pledge my honor that I have abided by the Stevens Honor Code."

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NOTE 1: For the final you may use up to five (5) 8 1/2'' by 11'' sheets of notes that you have prepared in reviewing for this test. You MAY NOT use review sheets that have been prepared by others in the class.

NOTE 2: This quiz counts for 20% of your grade for the course. Other components of the final grade are the three Case Studies (20% each), and 20% for homework and class participation.

PROBLEM 1. (15 points)

a) __X__ In accord with the Stevens Honor code, I will complete (or have already completed) the online course evaluation for ME 345 at http://www.stevens.edu/assess. (Check to confirm.) (3 points)

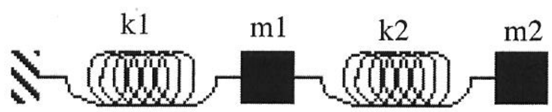
3 points whether checked or not.

all = 6
1/2 = 4

b) Briefly discuss the utility of analyzing the maximum von Mises stress in the context of a stress analysis. (6 points)

Max. VM stress gives a scalar stress measure to compare to, for instance, the yield stress of a material. Useful for analysis of 2D and 3D stress states.

c) For a particular two mass - two spring system, the state equations for the velocities of the masses as a function of time were found to be:



$$\underline{v_m} = \begin{Bmatrix} v_{m1}(t) \\ v_{m2}(t) \end{Bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(a_1 \cos(t) + b_1 \sin(t)\right) + \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \left(a_2 \cos(\sqrt{6}t) + b_2 \sin(\sqrt{6}t)\right)$$

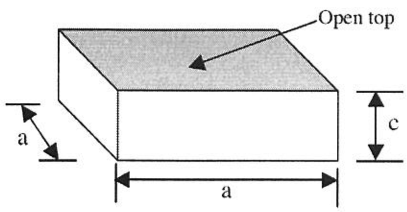
Find one set of values for a_2 and b_2 that will result in a normal mode behavior of the system. In this case, if the velocity of mass 1 is –5m/s (i.e. v_{m1}=-5m/s), what will the velocity of mass 2 be? (6 points)

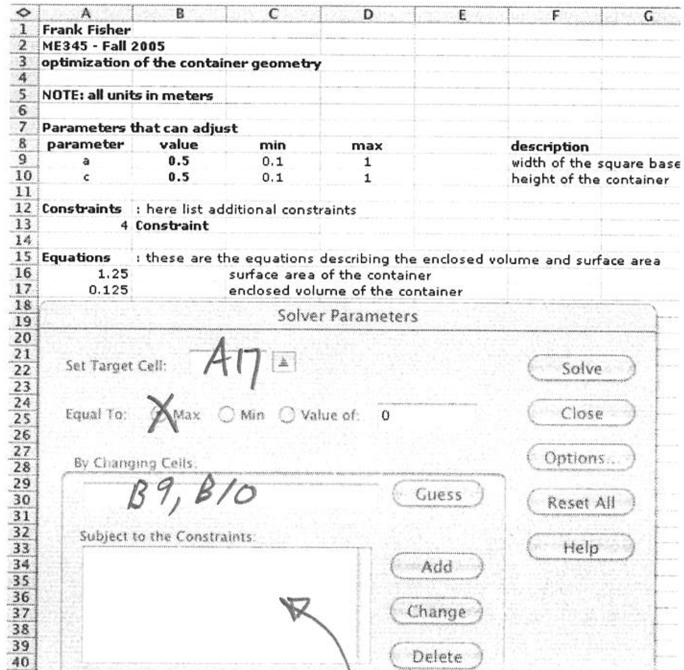
$a_2 = 0,\ b_2 = 0 \Rightarrow \underset{\sim}{v_m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (a_1 \cos t + b_1 \sin t)$

FOR ALL t, in this mode $v_{m2} = 2 v_{m1}$. If $v_{m1} = -5$ m/s, then $v_{m2} = -10$ m/s.

PROBLEM 2. (18 points)

The geometry of the *open-top, square-base* container (shown in the Figure) with a base dimension of a and height c is to be optimized to provide **maximum volume** while having a surface area less than or equal to 4 m^2. The range of variables for a and c are to be between 0.1 and 1 m.





◇	A	B	C	D	E	F	G
1	Frank Fisher						
2	ME345 - Fall 2005						
3	optimization of the container geometry						
4							
5	NOTE: all units in meters						
6							
7	Parameters that can adjust						
8	parameter	value	min	max		description	
9	a	0.5	0.1	1		width of the square base	
10	c	0.5	0.1	1		height of the container	
11							
12	Constraints	: here list additional constraints					
13	4	Constraint					
14							
15	Equations	: these are the equations describing the enclosed volume and surface area					
16	1.25		surface area of the container				
17	0.125		enclosed volume of the container				

Solver Parameters

Set Target Cell: A17

Equal To: X Max ○ Min ○ Value of: 0

By Changing Cells: B9, B10

Subject to the Constraints:

Solve | Close | Options... | Reset All | Help | Guess | Add | Change | Delete

a) Identify and derive any equations necessary to solve this optimization problem using the Excel solver tool. (6 points)

$SA = 4(a)(c) + a^2$

$V = a^2 c$

b) Clearly describe in words (or alternatively using the Figure on the right) how to solve this multivariate problem using the Excel solver tool. Clearly identify where how the equations above are used within the spreadsheet. (6 points)

A16 → put eqn. for SA above

A17 → put eqn. for V above

For rest, see figure

B9 > C9
B9 < D9
B10 > C10
B10 < D10
A16 < A13

(note: EXCEL SOLUTION : $a = 1$, $c = .75$, $V = .75$)

c) Neglecting the constraints on the values for a and c above, one **could** solve analytically for the value of a that satisfies the optimization problem (do NOT solve). Will the optimized volume in this case (found analytically) be greater than, less than, or equal to that found through the Excel solution in Part B above? Why? (6 points)

3

only talk about V

The constraints on "a" and "c" mean that the optimal volume in EXCEL will be $\leq V_{nc}$ (no constraints).

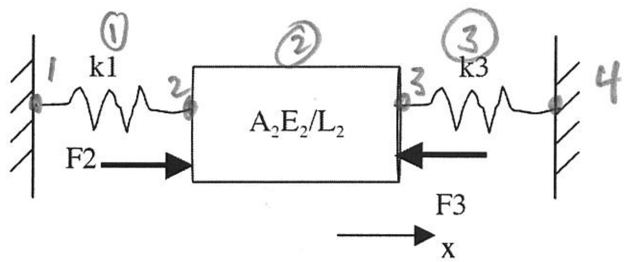
Analytical solution (not necessary):

$SA = 4 = 4ac + a^2 \Rightarrow c = \frac{a}{4}(4 - a^2)$. Substitute into $V = a^2 c \Rightarrow V = a - \frac{a^3}{4}$. Set $\frac{\partial V}{\partial a} = 0 \Rightarrow a = \sqrt{4/3}$, $c = .577$.

PROBLEM 3. (26 points)

Consider the **1D problem below (x-direction)**, where gravity effects are negligible. Prior to the application of the external forces, springs 1 and 3 are at their unstretched (equilibrium) length. *Be consistent with units.*

In the figure below, k1 = 10 N/mm, k3 = 20 N/mm, and A_2E_2/L_2=50 N/mm, whereas the magnitudes of the forces F2 and F3 are 100 N and 90 N *in the directions shown in the Figure*, respectively.



Recall that the element stiffnesses for a spring element and a rod element are as given below:

spring

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_I \\ d_J \end{Bmatrix} = \begin{Bmatrix} F_I \\ F_J \end{Bmatrix}$$

rod

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_I \\ d_J \end{Bmatrix} = \begin{Bmatrix} F_I \\ F_J \end{Bmatrix}$$

a. Clearly identify the nodes and elements on the Figure above. (1 point)

b. Assemble and <u>clearly label</u> the global stiffness matrix. (7 points)

Ele 1: $\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$ Ele 2: $\begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix} = \frac{A_2E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix}$

Ele 3: $\begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} = k_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_3 \\ d_4 \end{Bmatrix}$

Assemble for global stiffness:

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & & \\ -k_1 & k_1 + \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} & \\ & -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} + k_3 & -k_3 \\ & & -k_3 & +k_3 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

c. Solve for the unknown nodal displacements in the problem. (7 points)

From Part b, enforce boundary conditions to get

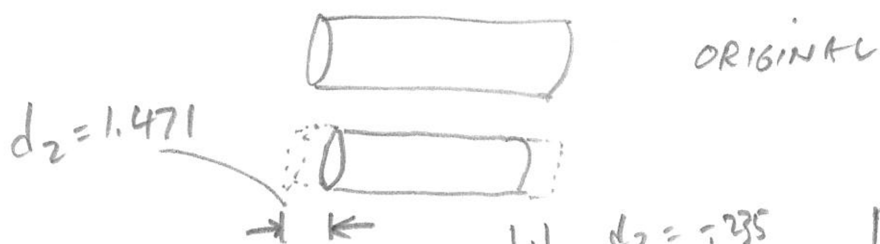
$$\begin{Bmatrix} \cancel{f_2}\ 100 \\ \cancel{f_3}\ -90 \end{Bmatrix} = \begin{bmatrix} 60 & -50 \\ -50 & 70 \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix}$$

$$\begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix} = \text{inv} \begin{bmatrix} 60 & -50 \\ -50 & 70 \end{bmatrix} \begin{Bmatrix} 100 \\ -90 \end{Bmatrix} \Rightarrow \boxed{\begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1.471 \\ -.235 \end{Bmatrix} mm}$$

d. Identify which, if any, elements are in compression. (4 points)

only element 2 (rod) in compression

e. What is the deformation of the rod due to the application of the applied forces? (4 points)



$$\delta = d_3 - d_2$$
$$= -.235 - 1.471$$
$$\boxed{\delta = -1.706 \text{ mm}}$$

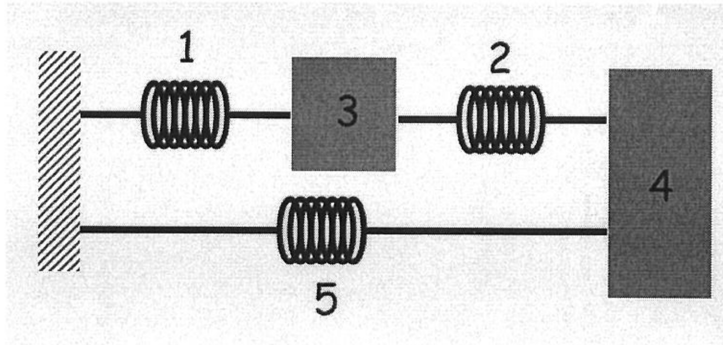
f. What is the internal force in Spring 1 due to the application of the applied forces? (4 points)

$f_1 = k_1 x_1$ — elongation of Spring 1.

$$f_1 = \left(110 \frac{N}{mm}\right)(1.471 mm) = \boxed{14.71 \text{ N}}$$

PROBLEM 4. (26 points)

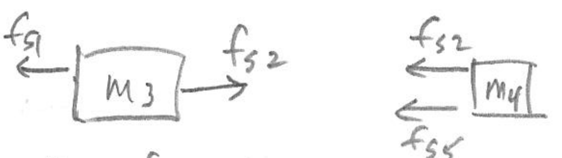
The one-dimensional problem shown at right consists of 3 springs (labeled 1, 2, and 5) and 2 masses (labeled 3 and 4). Standard notation and labeling of the system parameters (i.e. k_2 is the spring constant of spring 2) is assumed.



a. Derive the **first order** state equations for the system. Clear and legible work will be eligible for partial credit. (20 points)

1. $f_{s1} = k_1 x_{s1}$, $f_{s2} = k_2 x_{s2}$, $f_{s5} = k_5 x_{s5}$

2. $V_{s1} = V_{m3}$, $V_{s2} = V_{m4} - V_{m3}$, $V_{s5} = V_{m4}$

3. 

$-f_{s2} - f_{s5} = m_4 a_4$.

$-f_{s1} + f_{s2} = m_3 a_3$

4. SV's : x_{s1}, x_{s2}, x_{s5}, V_{m3}, V_{m4}

5. $x_{s1}' = V_{s1} = V_{m3}$ ✓

$x_{s2}' = V_{s2} = V_{m4} - V_{m3}$ ✓

$x_{s5}' = V_{s5} = V_{m4}$ ✓

$V_{m3}' = a_3 = \frac{1}{m_3}(-f_{s1} + f_{s2}) = \frac{1}{m_3}(-k_1 x_{s1} + k_2 x_{s2})$ ✓

$V_{m4}' = a_4 = \frac{1}{m_4}(-f_{s2} - f_{s5}) = \frac{1}{m_4}(-k_2 x_{s2} - k_5 x_{s5})$ ✓

Could also write in matrix form (not necessary)

$$\begin{Bmatrix} x_{s1}' \\ x_{s2}' \\ x_{s5}' \\ v_{m3}' \\ v_{m4}' \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ -k_1/m_3 & k_2/m_3 & 0 & 0 & 0 \\ 0 & -\frac{k_2}{m_4} & -\frac{k_5}{m_4} & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_{s1} \\ x_{s2} \\ x_{s5} \\ v_{m3} \\ v_{m4} \end{Bmatrix}$$

Convert to second order equation (not necessary)

$$v_{m3}'' = \frac{1}{m_3}(-k_1 v_{s1} + k_2 v_{s2}) = \frac{1}{m_3}\left[-k_1(v_{m3}) + k_2(v_{m4} - v_{m3})\right]$$

$$v_{m4}'' = \frac{1}{m_4}(-k_2 x_{s2} - k_5 x_{s5}) = \frac{1}{m_4}\left[-k_2(v_{m4} - v_{m3}) - k_5(v_{m4})\right]$$

write this in matrix form:

$$\begin{Bmatrix} v_{m3}'' \\ v_{m4}'' \end{Bmatrix} = \begin{bmatrix} -\frac{k_1}{m_3} - \frac{k_2}{m_3} & \frac{k_2}{m_3} \\ \frac{k_2}{m_4} & -\frac{k_2}{m_4} - \frac{k_5}{m_4} \end{bmatrix} \begin{Bmatrix} v_{m3} \\ v_{m4} \end{Bmatrix}$$

substitute system parameters (not necessary)

$$\begin{Bmatrix} v_{m3}'' \\ v_{m4}'' \end{Bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -3 \end{bmatrix} \begin{Bmatrix} v_{m3} \\ v_{m4} \end{Bmatrix}$$

Solve Eigenvalue problem (not necessary)

$$\det([A] - \lambda^2[I]) = 0 \Rightarrow \begin{bmatrix} -3-\lambda^2 & 2 \\ 1 & -3-\lambda^2 \end{bmatrix} = 0$$

$$\lambda^4 + 6\lambda^2 + 7 = 0$$

$$\lambda_1^2 = -3 + \sqrt{2} = -1.6$$

$$\lambda_2^2 = -3 - \sqrt{2} = -4.4$$

Find eigenvectors (not necessary)

$\lambda_1^2 = -1.6$

$$\begin{bmatrix} -1.4 & 2 \\ 1 & -1.4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 = 1,\ v_2 = .7$

$\lambda_2^2 = -4.4$

$$\begin{bmatrix} 1.4 & 2 \\ 1 & 1.4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 = 1,\ v_2 = -.7$

General solution (not necessary)

$$\underset{\sim}{v} = \begin{bmatrix} v_{BM3} \\ v_{m4} \end{bmatrix} = \begin{bmatrix} 1 \\ .7 \end{bmatrix} \left(a_1 \cos\sqrt{1.6}\,t + b_1 \sin\sqrt{1.6}\,t \right) + \begin{bmatrix} 1 \\ -.7 \end{bmatrix} \left(a_2 \cos\sqrt{4.4}\,t + b_2 \sin\sqrt{4.4}\,t \right)$$

Assuming that your first order equations above are correct, one *could* show (you do NOT need to do this here, although it would allow you to check your solution to the first part of the problem) that the equivalent second order system can be written in matrix form as:

$$\begin{Bmatrix} v_3'' \\ v_4'' \end{Bmatrix} = \begin{bmatrix} -\frac{(k_1+k_2)}{m_3} & \frac{k_2}{m_3} \\ \frac{k_2}{m_4} & -\frac{(k_2+k_5)}{m_4} \end{bmatrix} \begin{Bmatrix} v_3 \\ v_4 \end{Bmatrix}$$

Further assuming that k_1=1, k_2=2, k_5=4, m_3=1, and m_4=2; for a particular set of initial conditions one can show that the state equations for the system can be written as:

$$\underline{v_m} = \begin{Bmatrix} v_{m3}(t) \\ v_{m4}(t) \end{Bmatrix} = \begin{bmatrix} 1 \\ 0.707 \end{bmatrix} \left(\cos(\sqrt{1.6}t)\right) + \begin{bmatrix} 1 \\ -0.707 \end{bmatrix} \left(3\sin(\sqrt{4.4}t)\right)$$

b. List a ***complete set*** of initial conditions necessary to provide the system behavior given above. (6 points)

NOTE: Previous pages of derivations not necessary for this problem.

For each 2nd order equation, need two ICs → 4 ICs total.

velocity → $v_m = \begin{vmatrix} 1 \\ .707 \end{vmatrix} \cos\sqrt{1.6}\,t + \begin{vmatrix} 1 \\ -.707 \end{vmatrix} 3 \sin \sqrt{4.4}\,t$ (t=0)

at $t=0$, $\underline{v_m} = \begin{Bmatrix} v_{m3} \\ v_{m4} \end{Bmatrix} = \begin{vmatrix} 1 \\ .707 \end{vmatrix}(1) + \begin{vmatrix} 1 \\ -.707 \end{vmatrix}(0) = \underline{\begin{Bmatrix} 1 \\ .707 \end{Bmatrix}}$

$v_m' = a_m = \begin{vmatrix} 1 \\ .707 \end{vmatrix} \cancel{\sqrt{1.6}\,(-\sin\sqrt{1.6}\,t)} + \begin{vmatrix} 1 \\ -.707 \end{vmatrix} (3)(\sqrt{4.4}) \cos\sqrt{4.4}\,t$ (t=0)

$v_m'(t=0) = \underset{\sim}{a_m}(t=0) = \underline{\begin{Bmatrix} a_{m3}(t=0) \\ a_{m4}(t=0) \end{Bmatrix}} = \begin{vmatrix} 1 \\ -.707 \end{vmatrix} (3)(\sqrt{4.4}) = \underline{\begin{Bmatrix} 6.29 \\ -4.45 \end{Bmatrix}}$

⇒ at $t=0$, $v_{m3} = 1$, $v_{m4} = .707$, $a_{m3} = 6.29$, $a_{m4} = -4.45$ ⇐

PROBLEM 5. (15 points)

Consider the problem of a hanging rod subject to gravity as shown below. The rod has a density ρ, a modulus E, an original (undeformed) length L, and a constant cross-sectional area A. Assume standard metric units.

Due to gravity, the rod is subject to a distributed force along its length. However, it has been suggested that one can model this problem by simply treating the *entire* weight of the rod as an applied force at the end of the rod, i.e. $W_{PL} = \rho g V = \rho g A L$

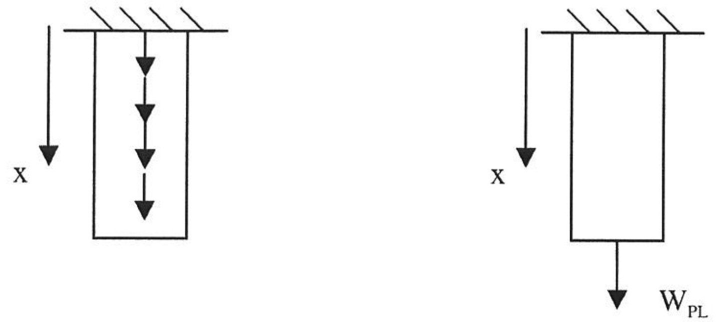
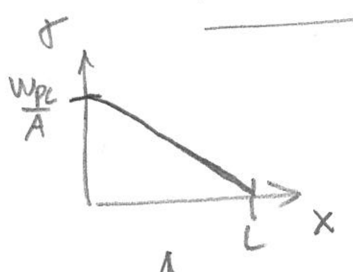


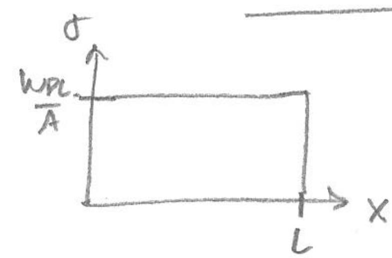
Figure. (left) distributed loading. (right) Weight modeled as a point load.

Discuss the implications of this assumption. How will the solution for the distributed loading case (left in the Figure above) differ from that of the solution for the point load case (right in the Figure above)? Discuss in terms of displacements and stresses in the rod, as well as comparison of the finite element solution to the exact solution in each case (assuming the use of linear, constant stress elements as we have used throughout the class).

DISTRIBUTED LOADING

POINT LOAD





actual case

approximation => overestimates the stress => overestimates the displacement.

FEM will approximate this, since the stress in an element is not uniform. As number of elements increases, better approx. of the exact solution.

FEM matches this w/ one element, since constant stress element.