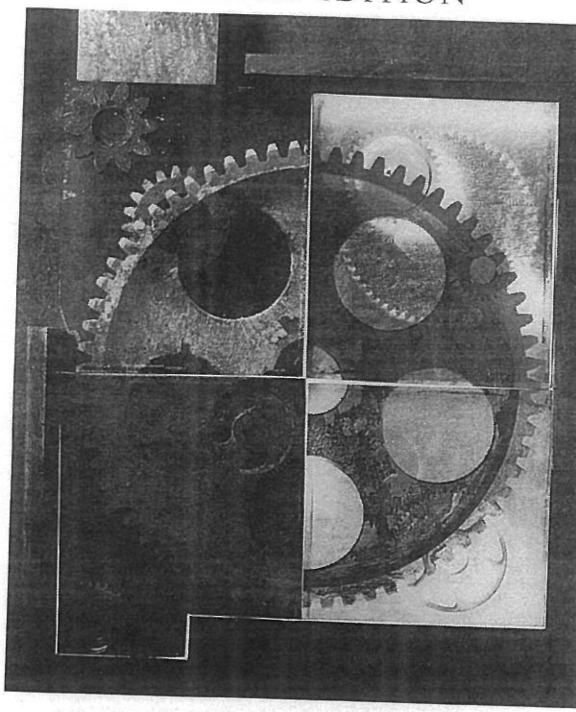
Design of MACHINE ELEMENTS

EIGHTH EDITION



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Sometimes only a portion of a stressed body is in compression, as, for example, the compression flange of a beam. The danger of buckling may be present here if sufficient lateral support is lacking. Such action has been the cause of failures and should be carefully guarded against by the designer[10].

If the load P approaches the critical load P_e, Eq. (36) indicates that the stress becomes infinite. Although there can be no such stress, it is characteristic of column equations to indicate the buckling phenomenon in this manner.

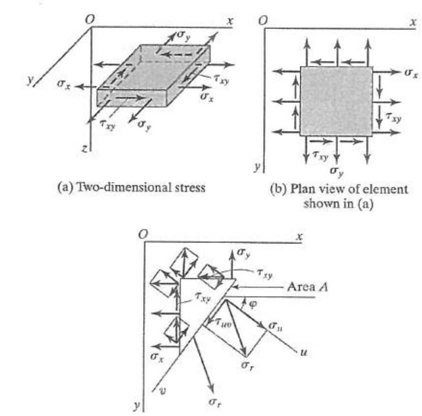
The hydraulic cylinder with extended piston rod forms a column[11].

START

1-19 STRESSES IN ANY GIVEN DIRECTION

The stresses in a body, as found by the equations of this chapter, have definite directions. It is sometimes necessary to have the stresses at directions other than those given by the equations.

Figure 1-30(a) shows an element of a plate with the vertical surfaces subjected to the general two-dimensional state of stress. The element has been cut from a larger plate so that stresses σ_x, σ_y, and τ_{xy} represent the effect of the surrounding



(a) Two-dimensional stress

(b) Plan view of element shown in (a)

(c) Components of stress in directions u and v

Figure 1-30 Shear and normal stress on element at any angle φ.

[10] A complete treatment of this subject may be found in the reference works at the end of the chapter.

[11] For design equations, see p. 136 of Reference 3, end of chapter.

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material on the element. A plan view of the element is shown in Fig. 1-30(b). Suppose stresses σ_x, σ_y, and τ_{xy} are known, and that it is necessary to find the values of the stresses on an inclined surface whose normal makes an angle φ with the x axis as shown in Fig. 1-30(c). Angle φ is an arbitrarily chosen angle and determines the directions of the u and v axes.

Assume that stress σ_r must be applied to the cut surface in order to maintain equilibrium of the remaining portion of the plate. Resultant stress σ_r can be resolved into the components of normal stress σ_u and shear stress τ_{uv} as shown.

If the area of the inclined surface is A, then the area of the horizontal side of the body will be $A \sin\varphi$, and the area of the vertical side, $A \cos\varphi$. Since the plate of Fig. 1-30(c) is in equilibrium, the projections of the forces on the perpendicular to the inclined surface must be in equilibrium. Multiplication of stress by area and then by the appropriate trigonometric function gives the following equation for σ_u:

$$\sigma_u = 2\tau_{xy}\sin\varphi\cos\varphi + \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi$$

The trigonometric terms should be changed by the substitution of the equations involving the double angles. Then,

$$\sigma_u = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi + \tau_{xy}\sin 2\varphi \tag{37}$$

If the element in Fig. 1-30 is cut at 90° to the direction in sketch (c), summation of the forces will give the equation for the normal stress in the v direction:

$$\sigma_v = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi - \tau_{xy}\sin 2\varphi \tag{38}$$

Thus, the normal stresses in the material at any desired angle φ can be found by use of the above equations. Should the equation give a negative result, the corresponding stress is compressive.

In a similar manner, τ_{uv} can be found by making the sum of the projections of all forces parallel to the cut surface equal to zero. Hence,

$$\tau_{uv} = \tau_{xy}(\cos^2\phi - \sin^{2\phi}) - (\sigma_x - \sigma_y)\sin\varphi\cos\varphi$$

or

$$\tau_{uv} = \tau_{xy}\cos 2\varphi - \frac{\sigma_x - \sigma_y}{2}\sin 2\varphi \tag{39}$$

The shear stress τ_{uv} at any desired angle φ can thus be found by Eq. (39). A positive result for τ_{uv} means that the stress is directed as in Fig. 1-30(c), and a negative result means that the stress is directed oppositely.

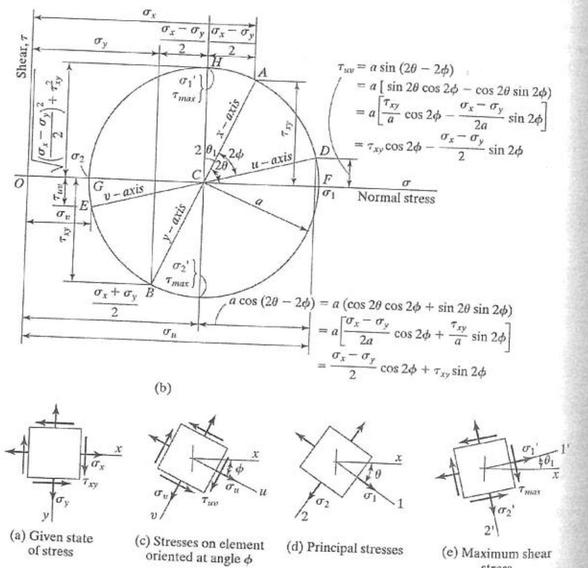
Angle φ is positive when taken clockwise from the x axis.

1-20 THE MOHR CIRCLE

A graphical solution to the combined stress problem known as the Mohr circle, will now be given. Use of this method rather than the previously derived equations usually effects a considerable saving in time. However, certain conventions regarding signs and directions must be understood and carefully followed.

Figure 1-31 shows the perpendicular axes σ and τ. Normal stresses, regardless of the inclination of the surface on which they act, are plotted horizontally positive, or tension, to the right of the origin, and negative, or compression, to the left. Shear stresses are plotted vertically upward or downward on the diagram. The normal and shear stresses at a point in the body, thus, become the coordinates of a point on the circle.

Stresses σ_x and τ_{xy} acting on the right and left edges of the plate in Fig. 1-30(b) locate point A in Fig. 1-31. Tension σ_x is plotted to the right in accordance with the



(a) Given state of stress (c) Stresses on element oriented at angle ϕ (d) Principal stresses (e) Maximum shear stress

Figure 1-31 Mohr circle for two-dimensional stress.

previously mentioned rule. Since shear stress τ_{xy} tends to rotate the element in a clockwise direction, it is plotted upward. Stresses σ_x and τ_{xy} of the upper and lower edges of the plate shown in Fig. 1-30(b) locate point B in Fig. 1-31. Tension σ_y is plotted to the right. Since shear stress τ_{xy} on these surfaces tends to produce counterclockwise rotation, it is plotted downward. The Mohr circle is drawn with line AB as a diameter. Greater facility in the determination of angles will be obtained if radii AC and BC are marked x axis and y axis, respectively.

To find the stresses on an element oriented at angle φ, as shown in Fig. 1-31(c), the angle 2φ is laid off from CA in the same direction as angle φ is turned in the body. Diameter DE is thus located.

The horizontal projection of CD has the value shown in the figure. When this is added to OC, the result is the value of σ_u as given by Eq. (37). The vertical projection of CD has the value shown on the figure. This is equal to τ_{uv} as given by Eq. (39). It is plain that the coordinates of point D of the circle are equal to the normal and shear stresses as found by the combined stress equations.

Stresses σ_v and τ_{uv} for the surface in Fig. 1-31(c), whose normal lies at angle $(90^\circ + \varphi)$ from the x axis, are given by the coordinates of point E.

A clockwise angle φ on the body corresponds to a clockwise angle of 2φ on the circle, and vice versa.

Values of stresses σ_u, σ_v, and τ_{uv} change as angle φ is changed. The maximum and minimum values of the normal stresses are called the principal stresses, and are designated σ_1 and σ_2 respectively. Their values can be found from the abscissas for points F and G in Fig. 1-31(b). The element for the principal stresses is oriented at angle θ to the x axis as shown in Fig. 1-31(d). As shown by the circle, the value of θ can be found by the following equation:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}, \qquad \text{for principal stresses} \tag{40}$$

The radius of the circle has the value shown. The equations for σ_1 and σ_2 are as follows:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_2 - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{41}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_2 - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{42}$$

It should be noted that the sides of the element for principal stresses are free from shearing stress. If shear stress τ_{xy} should be equal to zero, stresses σ_x and σ_y would become the principal stresses σ_1 and σ_2[12].

The maximum shearing stress to which the material is subjected has a value equal to the radius of the circle. On the circle, point H is located 90° from points F and G for principal stresses. In the body, the surfaces for maximum shear stress are thus inclined 45° to the surfaces for the principal stresses. The element of maximum

[12] When the stress consists only of simple tension, it can be designated σ rather than σ_1.

shearing stress, as shown in Fig. 1-31(e), is inclined at θ to the x axis. As shown by the circle, the value of θ can be found by the following equation:

$$\tan 2\theta_1 = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \tag{43}$$

The value of the maximum shearing stress is

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{x}\right)^2 + \tau_{xy}^2} \tag{44}$$

The circle of Fig. 1-31 indicates that at points of maximum shear, such as at H, normal stresses σ_1 and σ_2 are present whose value is given by the equation

$$\sigma_1' = \sigma_2' = \frac{\sigma_x + \sigma_y}{2} \tag{45}$$

When shear stress τ is equal to zero, the radius of the circle or the maximum shearing stress is equal to

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2) \tag{46}$$

EXAMPLE 1-15

Problem Statement: Let the state of stress at some point in a body be defined as follows:

$$\sigma_x = 20{,}000 \text{ psi}, \qquad \sigma_y = -4000 \text{ psi}, \qquad \tau_{xy} = 5000 \text{ psi}$$

(a) Draw the view of the element for the given state of stress and mark values thereon.
(b) Draw the Mohr circle for the given state of stress and mark completely.
(c) Draw the element oriented 30° clockwise from the x-axis and show values of all stresses.
(d) Draw the element correctly oriented for principal stresses and show values.
(e) Draw the element for maximum shear stress and mark values of all stresses

Given Information: The three stress components.

Assumptions: The stresses are in a single plane and are not three-dimensional.

Solution Method: Mohr's Circle.

Solution Details: (a), (b). The given state of stress and the Mohr circle are shown in Fig. 1-32(a) and (b), respectively.

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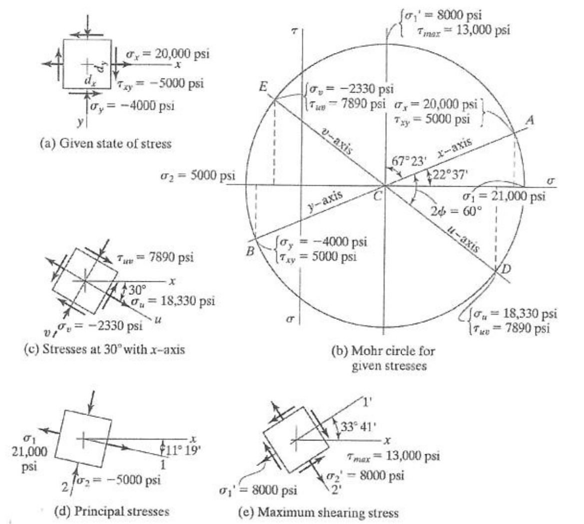

Figure 1-32 Solution of Example 1-15 by Mohr circle.

(c) Diameter ECD should be drawn at 60° clockwise to the x-axis of the circle, and stresses σ_u and σ_v should be scaled and placed on the element of Fig. 1-32(c). Since point D lies below the u-axis, shear stress τ_{uv} crosses the u-axis of Fig. 1-32(c) in the direction that causes a counterclockwise moment on the element. Likewise, since E lies above the σ-axis, stress τ_{uv} crosses the v-axis in sketch (c) in the direction that causes a clockwise moment on the element.

(d) Principal stresses σ_1 and σ_2, together with their angle of inclination, are scaled directly from the circle, and are shown acting on an element properly oriented in Fig. 1-32(d).

(e) The maximum shear stress z_{max} and the corresponding normal stress σ_1 are shown on the element of Fig. 1-32(e). The arrows are directed in accordance with the previously explained rules.

The advantages of the graphical method for solving combined stress problems should now be apparent. Not only is the method more rapid, but the state of stress for any

FAILURE THEORIES

to sliding is smaller than its resistance to separation. Failure takes place by yielding. Many ductile materials share the same yield point for compression as for tension.

A brittle material is one whose resistance to separation is less than its resistance to sliding. Failure takes place by fracture. A limit of about 5% elongation is usually taken as the dividing line between ductile materials and brittle materials. Most brittle materials have a considerably higher value for the ultimate strength in compression than for tension.

Under certain conditions, a material ordinarily said to be ductile will undergo a fracture or separation failure similar to that of a brittle material. Some of these conditions are (a) cyclic loading at normal temperatures (fatigue); (b) long-time static loading at elevated temperatures (creep); (c) impact or very rapidly applied loading, especially at low temperatures; (d) work hardening by a sufficient amount of yielding; (e) severe quenching in heat treatment if not followed by tempering; and (f) a three-dimensional state of stress in which sliding is prevented.

START HERE

~~2-3~~ PHENOMENOLOGICAL FAILURE THEORIES BASED ON STRESS

All phenomenological failure theories for static stress are based on the use of a uniaxial tensile or compression test as the simple test.

1. Maximum Normal Stress Theory of Failure

The hypothesis for the maximum normal stress theory of failure is that failure will occur in a complex part if any of the principal normal stresses exceeds the principal normal stress that gave rise to failure in the simple, uniaxial test. This can be stated in the following way:

$$S_{ypc} \le S_1 \le S_{ypt}$$
$$S_{ypc} \le S_2 \le S_{ypt}$$
$$S_{ypc} \le S_3 \le S_{ypt}$$

These failure equations can be converted into design equations by applying a factor of safety N_{fs} to the yield point stresses to get

IMPORTANT

$$\left.\begin{aligned} \frac{S_{ypc}}{N_{fs}} &\le S_1 \le \frac{S_{ypt}}{N_{fs}} \\ \frac{S_{ypc}}{N_{fs}} &\le S_2 \le \frac{S_{ypt}}{N_{fs}} \\ \frac{S_{ypc}}{N_{fs}} &\le S_3 \le \frac{S_{ypt}}{N_{fs}} \end{aligned}\right\} \qquad (1)$$

If this theory is applied to brittle materials, the ultimate stresses σ_{uc} and σ_{ut} are substituted for the yield point stresses in the equations. This failure theory can be

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implemented using spreadsheet Module 2-1, which allows the user to insert the principal normal stresses and check the inequalities.

2. Maximum Shear Stress Theory of Failure

The hypothesis for the maximum shear stress theory of failure is that failure will occur in a complex part if any of the principal shear stresses exceeds the principal shear stress that gave rise to failure in the simple, uniaxial test. Since the shear stress at failure for uniaxial tension is one-half of the normal yield point stress, this failure theory can be stated in mathematical terms as

$$-S_{yp} \le (S_1 - S_2) \le S_{yp}$$
$$-S_{yp} \le (S_2 - S_3) \le S_{yp}$$
$$-S_{yp} \le (S_3 - S_1) \le S_{yp}$$

These failure equations can be converted into design equations by applying a factor of safety to get

IMPORTANT

$$\left\{\begin{aligned} \frac{-S_{yp}}{N_{fs}} &\le (S_1 - S_2) \le \frac{S_{yp}}{N_{fs}} \\ \frac{-S_{yp}}{N_{fs}} &\le (S_2 - S_3) \le \frac{S_{yp}}{N_{fs}} \\ \frac{-S_{yp}}{N_{fs}} &\le (S_3 - S_1) \le \frac{S_{yp}}{N_{fs}} \end{aligned}\right. \tag{2}$$

This failure theory can be implemented using spreadsheet Module 2-2 which allows the user to insert the principal normal stresses and check the inequalities.

Before proceeding with the development of other failure theories, let us look at an application using the two that we have already discovered.

EXAMPLE 2-1

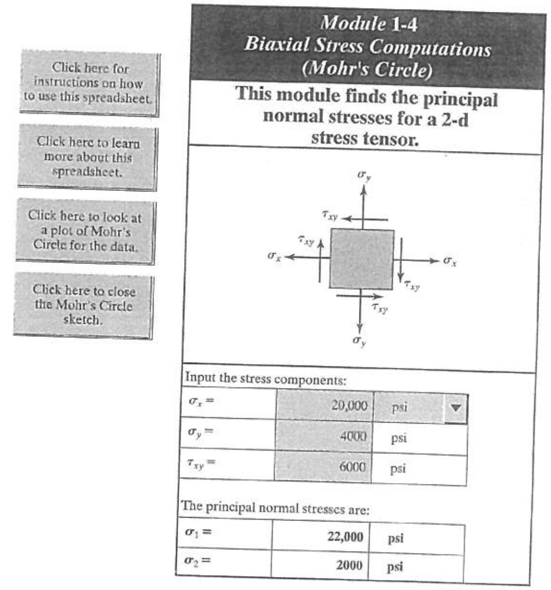
Problem Statement: The stresses at a point in a body are $\sigma_x = 20{,}000$ psi, $\sigma_y = 4000$ psi and $\tau_{xy} = 6000$ psi. If the factor of safety is to be $N_{fs} = 2.0$ and the material is No. 40 gray cast iron, determine if the design is safe by the maximum normal stress theory of failure.

Given Information: $\sigma_x = 20{,}000$ psi,
$\sigma_y = 4000$ psi,
$\tau_{xy} = 6000$ psi,
$N_{fs} = 2.0$ and
the material is No. 40 gray cast iron.

Assumptions: The stresses are biaxial and the material is such that the maximum normal stress theory of failure is applicable.

Solution Method: Module 1-4, Module 2-1A.
From Table 14-16 we obtain $\sigma_{uc} = -135{,}000$ psi and $\sigma_{ut} = 40{,}000$ psi for No. 40 gray cast iron.

Solution Details: To solve this problem we must first determine the principal normal stresses. Since this is a biaxial state of stress, we could use either Mohr's circle or Module 1-4 to determine these stresses. To use Module 1-4,



we need to insert the appropriate stress values into the shaded cells and the result will be provided in later cells as indicated.

Now that we have the principal normal stress values of 22,000 psi and 2000 psi, we can utilize the maximum normal stress theory of failure.

Click here for instructions on how to use this spreadsheet.

Click here to learn more about this spreadsheet.

Module 2-1a Maximum Normal Stress Theory of Failure (Brittle Materials)					
Input the principal normal stresses and their units:					
$S_1 =$	22,000	psi			
$S_2 =$	2000	psi			
$S_3 =$	0	psi			
Input the Yield point values and the Factor of Safety:					
$S_{uc} =$	−135,000	psi	(Compression Ultimate Strength)		
$S_{ut} =$	40,000	psi	(Tension Ultimate Strength)		
$N_{fs} =$	2.00		(Factor of Safety)		
Check each of the three equations of the failure theory:					
$\frac{S_{uc}}{N_{fs}} \le S_1 \le \frac{S_{ut}}{N_{fs}}$					
−67,500	≤	**22,000**	≤	**20,000**	**Fails**
$\frac{S_{uc}}{N_{fs}} \le S_2 \le \frac{S_{ut}}{N_{fs}}$					
−67,500	≤	**2000**	≤	**20,000**	**Safe**
$\frac{S_{uc}}{N_{fs}} \le S_3 \le \frac{S_{ut}}{N_{fs}}$					
−67,500	≤	**0**	≤	**20,000**	**Safe**

Since the material is No. 40 gray cast iron, we will use Module 2-1a, which is for brittle materials. If we insert the appropriate values and the principal normal stresses into Module 2-1a, the result will be as shown.

Clearly this part fails since the first of the three design equations for the failure theory is not satisfied. This design could be made safe if the factor of safety were lowered or if a different material with a higher ultimate strength in tension were used.

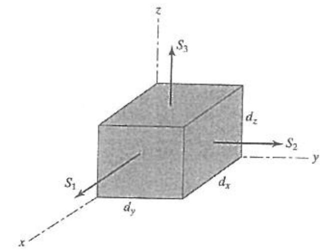


Figure 2-3 A rectangular block subjected to the principal normal stresses.

3 Maximum Strain Energy Theory of Failure

The hypothesis for the maximum strain energy theory of failure is that failure will occur in a complex part when the strain energy per unit volume exceeds that for a simple uniaxial tensile test at failure. To determine the strain energy per unit volume, we will need to look at a small rectangular block of the material that is "dx" wide, "dy" high and "dz" deep as shown in Fig. 2-3.

This block has the principal normal stresses applied to its faces as shown. The total strain energy will be the work done by the forces resulting from these stresses:

$$U = \text{Strain Energy} = \text{Work} = \int F\,\delta l \tag{3}$$

In this expression, δl is the distance moved, and F represents the force as the deflection occurs. The final forces generated by the stresses on each face will be the final stress on that face multiplied times the area of that face:

$$F_{y\,(\text{final})} = S_2\,dx\,dz$$
$$F_{x\,(\text{final})} = S_1\,dy\,dz$$
$$F_{z\,(\text{final})} = S_3\,dx\,dy$$

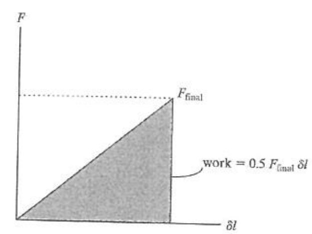


Figure 2-4 The linear relationship between load and deflection for the block.

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Since the stretching process is a linear one, the integral will be the area under the curve of F - versus - δl as shown in Fig. 2-4. Thus, the total strain energy will be

$$U = \text{Strain Energy} = \frac{F_{x(\text{final})}\delta_x}{2} + \frac{F_{y(\text{final})}\delta_y}{2} + \frac{F_{z(\text{final})}\delta_z}{2} \tag{4}$$

The distance moved for each direction will be the strain per unit length "ε" times the length, or

$$\delta_x = \varepsilon_1\, dx$$
$$\delta_y = \varepsilon_2\, dy$$
$$\delta_z = \varepsilon_3\, dz$$

The strains can be eliminated from these equations by means of Hooke's law:

$$\varepsilon_1 = \frac{1}{E}(S_1 - \mu S_2 - \mu S_3)$$
$$\varepsilon_2 = \frac{1}{E}(S_2 - \mu S_1 - \mu S_3)$$
$$\varepsilon_3 = \frac{1}{E}(S_3 - \mu S_1 - \mu S_2)$$

Adding together all three work components gives

$$U = \frac{dx\,dy\,dz}{2E}(S_1^2 + S_2^2 + S_3^2 - 2\mu(S_1S_2 + S_2S_3 + S_1S_3))$$

The strain energy per unit volume, u, will be the total strain energy, U, divided by the volume of the small block ($V = dx\,dy\,dz$):

$$u = \frac{1}{2E}(S_1^2 + S_2^2 + S_3^2 - 2\mu(S_1S_2 + S_2S_3 + S_1S_3))$$

The strain energy per unit volume at failure for the uniaxial tensile test will be

$$u = \frac{1}{2E}(S_{yp}^2)$$

Setting this as a limit for the complex case gives a failure equation of the form

$$(S_1^2 + S_2^2 + S_3^2 - 2\mu(S_1S_2 + S_2S_3 + S_1S_3)) \le S_{yp}^2$$

This equation can be converted to a design equation by applying a factor of safety to the yield point stress to get

IMPORTANT

$$\boxed{(S_1^2 + S_2^2 + S_3^2 - 2\mu(S_1S_2 + S_2S_3 + S_1S_3)) \le \left(\frac{S_{yp}}{N_{fs}}\right)^2} \tag{5}$$

This equation can be implemented by means of Module 2-4 in the spreadsheet package.

(see pg 137!)

4. Maximum Distortion Energy Theory of Failure d VON MISES u

The basis for the maximum distortion energy theory of failure is that the overall strain energy is composed of two parts. The first part is the energy associated with merely changing the volume of the part while the second part is associated with the distortion of the part. Thus, the total strain energy per unit volume "u" can be written as

$$u = u_v + u_d$$

where u_v is the energy of volume change per unit volume and u_d is the energy of distortion per unit volume. It is this distortion part of the strain energy that is the basis for this failure theory. The hypothesis for this theory is that failure will occur in the complex part when the distortion energy per unit volume exceeds that for a simple uniaxial tensile test at failure.

For purposes of describing this failure theory, the principal normal stresses can be thought of as being composed of two parts that are superimposed as shown in Fig. 2-5. For this superposition, the relationships will be

$$S_1 = S_1' + S_v$$
$$S_2 = S_2' + S_v$$
$$S_3 = S_3' + S_v$$

Here S_v represents the portion of the stress that causes volume change and the S_i' terms represent the portion of the principal normal stresses that cause distortion. If there is to be no volume change associated with the distortion components, then the sum of the strains arising from the distortion stresses must equal zero:

$$\varepsilon_1' + \varepsilon_2' + \varepsilon_3' = 0$$

We can write these strain components in terms of the stress components by means of Hooke's law:

$$\varepsilon_1' = \frac{1}{E}(S_1' - \mu S_2' - \mu S_3')$$
$$\varepsilon_2' = \frac{1}{E}(S_2' - \mu S_1' - \mu S_3')$$
$$\varepsilon_3' = \frac{1}{E}(S_3' - \mu S_1' - \mu S_2')$$

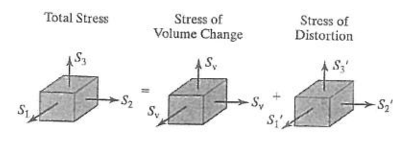

Figure 2-5 The component parts of the principal normal stresses.

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If we add together these three equations and set them equal to zero as required, the result will be

$$0 = S_1' + S_2' + S_3' - 2\mu(S_1' + S_2' + S_3')$$

This will be true for all values of μ if

$$0 = S_1' + S_2' + S_3'$$

If this relationship is used with the sum of the first three stress equations, the result will be

$$S_v = \frac{1}{3}(S_1 + S_2 + S_3)$$

This relationship can be used in the first three equations to express the distortion stresses in terms of only the principal normal stresses:

$$S_1' = \frac{2}{3}\left(S_1 - \frac{S_2}{2} - \frac{S_3}{2}\right)$$

$$S_2' = \frac{2}{3}\left(S_2 - \frac{S_1}{2} - \frac{S_3}{2}\right)$$

$$S_3' = \frac{2}{3}\left(S_3 - \frac{S_1}{2} - \frac{S_2}{2}\right)$$

The strain energy of volume change will be

$$u_v = 3\left(\frac{S_v \varepsilon_v}{2}\right)$$

Using Hooke's law of the form

$$\varepsilon_v = \frac{1}{E}(S_v - \mu S_v - \mu S_v) = \frac{S_v}{E}(1 - 2\mu)$$

Thus,

$$u_v = \left(\frac{1 - 2\mu}{6E}\right)(S_1 + S_2 + S_3)^2$$

We know that the distortion energy per unit volume is

$$u_d = u - u_v$$

We know that the total strain energy per unit volume was determined from the previous section to be

$$u = \frac{1}{2E}(S_1^2 + S_2^2 + S_3^2 - 2\mu(S_1S_2 + S_1S_3 + S_2S_3))$$

Taking the difference between this value and the change of volume strain energy per unit volume gives the final expression for the distortion energy per unit volume in terms of the principal normal stresses:

$$u_d = \left(\frac{1 + \mu}{3E}\right)(S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1)$$

Comparing this value with the value for a uniaxial case gives a failure equation of the form

$$(S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1) \le S_{yp}^2$$

This failure equation can be converted to a design equation by introducing the factor of safety associated with the yield stress. The result will be

important

$$\boxed{(S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1) \le \left(\frac{S_{yp}}{N_{fs}}\right)^2} \qquad (6)$$

This equation can be implemented by means of Module 2-4 in the spreadsheet package. Unlike the maximum strain energy theory of failure, this equation does not depend in any way on Poisson's ratio μ. This failure theory is also known as the shear–energy theory or the von Mises–Hencky theory. Before summarizing and comparing the design equations, let us look at an example problem utilizing this failure theory.

EXAMPLE 2-2

Problem Statement: Suppose that a triaxial state of stress is given by the following stress tensor:

$$\sigma_{ij} = \begin{bmatrix} 70 & 30 & 25 \\ 30 & 80 & 40 \\ 25 & 40 & 90 \end{bmatrix} \text{MPa}$$

If the material is steel with $S_{yp} = 386$ MPa and $N_{fs} = 3.0$, determine if the design is safe by the maximum distortion energy theory of failure.

Given Information: The six unique stress components of the stress tensor, $S_{yp} = 386$ MPa, and $N_{fs} = 3.0$

Assumptions: The stresses are three-dimensional and the material is ductile so that the maximum distortion energy theory of failure is applicable.

Solution Method: Module 1-5, Module 2-4.

Solution Details: The first step in the solution of this problem is to determine the principal normal stress components. This can be done using Module 1-5 in the spreadsheet package as shown next.

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Failure Theories Chap. 2

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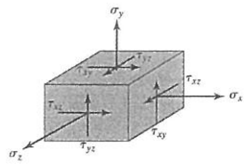
Click here for instructions on how to use this spreadsheet.

Click here to learn more about this spreadsheet.

Module 1-5
Triaxial Stress Computations

This module finds the principal normal stresses for a 3-d stress tensor.



Input the stress components:

σ_x =	70	MPa
σ_y =	80	MPa
σ_z =	90	MPa
τ_{xy} =	30	MPa
τ_{xz} =	25	MPa
τ_{yz} =	40	MPa

The principal normal stresses are:

S_1 =	41.42	MPa
S_2 =	145.21	MPa
S_3 =	53.37	MPa

Now that we have the principal normal stresses, we can utilize the distortion energy theory of failure to see if the part will fail. To do this we can fit the three principal normal stresses into Module 2-4 from the spreadsheet package.

Click here for instructions on how to use this spreadsheet.

Click here to learn more about this spreadsheet.

Module 2-4
Maximum Distortion Energy Theory of Failure

$$(S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_1S_3 - S_2S_3) \leq \left(\frac{S_{yp}}{N_{fs}}\right)^2$$

Input the principal normal stresses and their units:

S_1 =	41.42	MPa
S_2 =	145.21	MPa
S_3 =	53.37	MPa

Input the Yield point and the Factor of Safety:

S_{yp} =	386	MPa	(Yield Strength)
N_{fs} =	3.00		(Factor of Safety)

The result from the failure equation is:

Left-hand side =	9.67E+03
Right-hand side =	1.66E+04
Status =	Safe

As the spreadsheet indicates, the part is safe by the maximum distortion energy theory of failure. It is interesting to note that the part would not be safe by the maximum normal stress theory of failure since the largest of the principal normal stresses is sufficient to cause failure by this failure theory. This situation suggests that we should look closely at the accuracy of the various failure theories and consider carefully when each is most appropriately applied. This will be the topic we will consider next.

A Comparison of Failure Theories

Table 2-4 provides a summary of the attributes of the four failure theories to enable a designer to choose the best theory for a particular situation. This table indicates when a particular theory is applicable and provides an indication of why one theory would be preferred over another. For example, if the material being considered for design is brittle, the maximum normal stress theory is the proper choice. For ductile materials, the choice of theory will depend on the level of accuracy required and on the degree of computational difficulty that the user is willing to invest in the process.

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2-4 STRESS CONCEN

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TABLE 2-4 A COMPARISON OF THE ATTRIBUTES OF VARIOUS FAILURE THEORIES.

Failure Theory	Attributes
Maximum Normal Stress	The only theory for brittle materials.
Maximum Shear Stress	Good for ductile materials. Gives satisfactory results and is easy to use.
Maximum Strain Energy	Good for ductile materials. Gives better results but is more difficult to use. Requires the use of Poisson's ratio.
Maximum Distortion Energy	Good for ductile materials. Gives the best results. Easier to apply than maximum strain energy.

VON MISES

For ductile materials, the most accurate method is the maximum distortion energy theory of failure and the method that is easiest to apply is the maximum shear stress theory. All of these theories are based on the application of steady loads to parts that are shaped in ways that do not concentrate or amplify the resulting stress patterns. These topics will be the subject of the next two sections.

2-4 STRESS CONCENTRATION CAUSED BY SUDDEN CHANGE IN FORM

Abrupt changes in geometry can give rise to stress values that are larger than would be expected. This can be a source of difficulty for designers. Consider, for example, the state of stress in the tension member of two widths illustrated in Fig. 2-6. Near each end of the bar the internal force is uniformly distributed over the cross sections. The nominal stress in the right portion can be found by dividing the total load by the smaller cross-sectional area; the stress in the left portion can be found by dividing by the larger area. However, in the region where the width is changing, a redistribution of the force within the bar must take place. In this portion, the load is no longer uniform at all points on a cross section, but the material in the neighborhood of points B in Fig. 2-6 is stressed considerably higher than the average value. The stress situation is thus more complicated, and the elementary equation P/A is no longer valid. The maximum stress occurs at some point on the fillet, as at B, and is directed parallel to the boundary at that point.

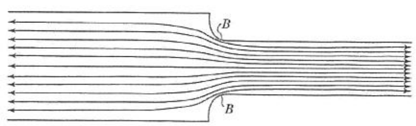


Figure 2-6 Stress concentration caused by a sudden change in cross section.

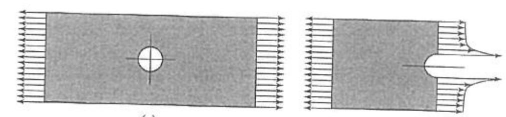


Figure 2-7 Stress concentration for a bar with a hole loaded in tension.

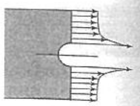
Another example is a bar in tension with a circular hole as shown in Fig. 2-7(a). If the bar is cut on the cross section of the hole, the tension stresses will be as shown in Fig. 2-7(b). The stress distribution along the cut surface is practically uniform until the neighborhood of the hole is reached, where it suddenly increases. This irregularity in the stress distribution caused by abrupt changes of form is called *stress concentration*. It occurs for all kinds of stress, axial, bending, or shear in the presence of fillets, holes, notches, keyways, splines, tool marks, or accidental scratches. Inclusions and flaws in the material or on the surface also serve as stress raisers. The maximum value of the stress at such points is found by multiplying the nominal stress as given by the elementary equation by a stress concentration factor K, or K_t, which is defined as follows:

$$K = \frac{\text{highest value of actual stress on fillet, notch, hole, etc.}}{\text{nominal stress as given by elementary equation for minimum cross section}} \tag{7}$$

Values of stress concentration factors can be found experimentally by photoelastic analysis or direct strain gage measurement. They can also be found by computer methods using finite element analysis.

Stress Concentration Factors

Stress concentration factors have been determined for a wide variety of geometric shapes and types of loading. The best known summary of results for various geometric shapes is the work by Peterson and is based on results from photoelastic testing done prior to 1951. More recently, researchers have developed mathematical models to approximate this classical data. Some of the best examples of these approximating models are documented by Norton, Pikley, and Young. In general, a stress concentration factor is applied to the stress computed for the net or smallest cross section. In this text, a number of spreadsheet modules have been provided to assist the designer with the determination of stress concentration factors for various geometric configurations involving abrupt changes in geometry. These spreadsheets utilize some of the models of Norton and also some linear interpolation from the data provided in the work of Peterson. These modules should enable the designer to find quickly the necessary concentration factors for a variety of geometric conditions. In many cases, the modules also provide information about the nominal stress and the actual stress using the concentration factor. Figures 2-8 through 2-21 show the stress concentration factors reported by Peterson and illustrate the fact that, Lots of examples → See pg 143



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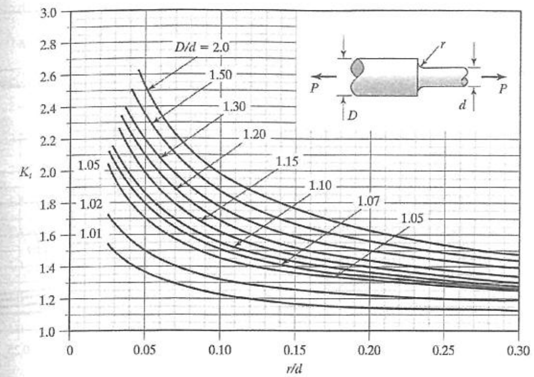


Figure 2-8 Stress concentration factor for a shaft with a shoulder fillet in axial tension. (Spreadsheet Module 2-5.)
(Curves from Peterson, R. E. "Design Factors for Stress Concentration, Parts 1 to 5," *Machine Design*, February–July, 1951.)

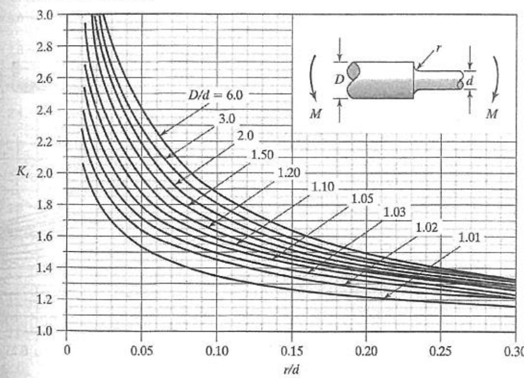


Figure 2-9 Stress concentration factor for a shaft with a shoulder fillet in bending. (Spreadsheet Module 2-6.)
(Curves from Peterson, R. E. "Design Factors for Stress Concentration, Parts 1 to 5," *Machine Design*, February–July, 1951.)

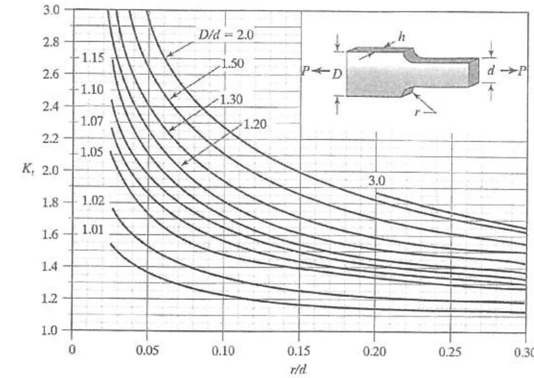


Figure 2-16 Stress concentration factor for a flat bar with a fillet in axial tension. (Spreadsheet Module 2-13.)
(Curves from Peterson, R. E. "Design Factors for Stress Concentration, Parts 1 to 5," *Machine Design*, February–July, 1951.)

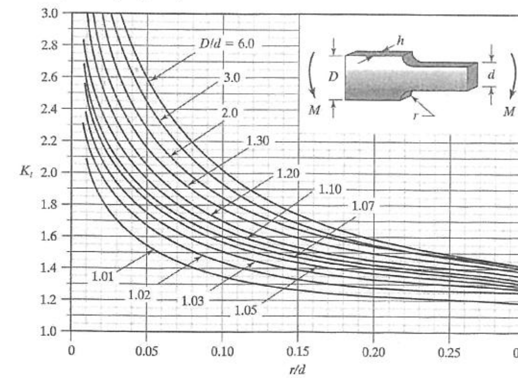


Figure 2-17 Stress concentration factor for a flat bar with a fillet in bending. (Spreadsheet Module 2-14.)
(Curves from Peterson, R. E. "Design Factors for Stress Concentration, Parts 1 to 5," *Machine Design*, February–July, 1951.)

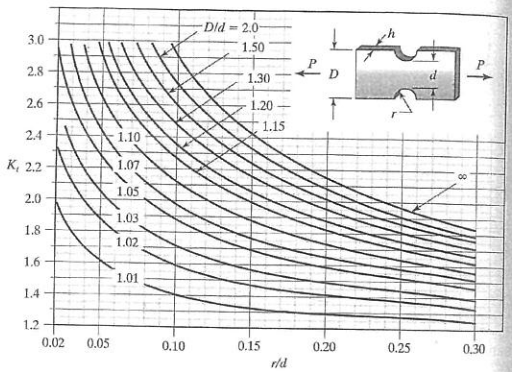


Figure 2-18 Stress concentration factor for a flat bar with a notch in axial tension. (Spreadsheet Module 2-15.)

(Curves from Peterson, R. E. "Design Factors for Stress Concentration, Parts 1 to 5," *Machine Design*, February–July, 1951.)

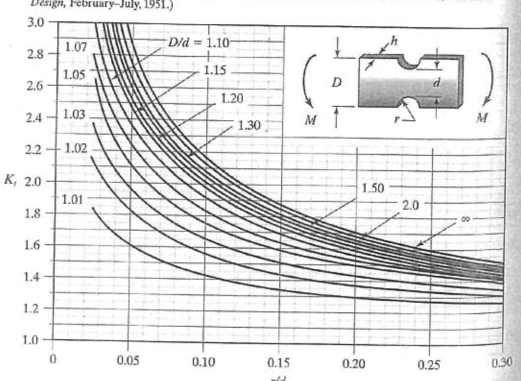
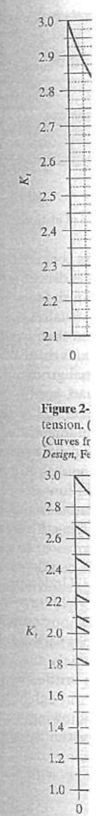


Figure 2-19 Stress concentration factor for a flat bar with a notch in bending. (Spreadsheet Module 2-16.)

(Curves from Peterson, R. E. "Design Factors for Stress Concentration, Parts 1 to 5," *Machine Design*, February–July, 1951.)



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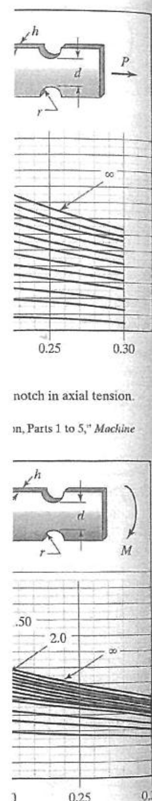

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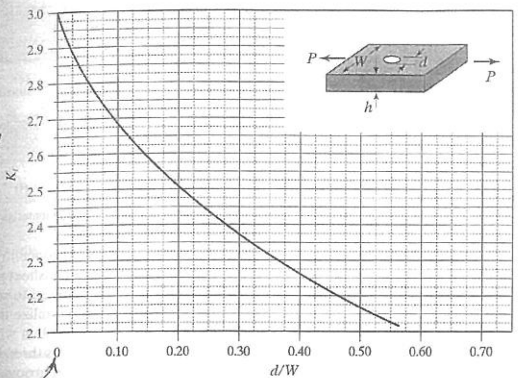

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Figure 2-20 Stress concentration factor for a flat bar with a transverse hole in axial tension. (Spreadsheet Module 2-17.)

(Curves from Peterson, R. E. "Design Factors for Stress Concentration, Parts 1 to 5," *Machine Design*, February–July, 1951.)

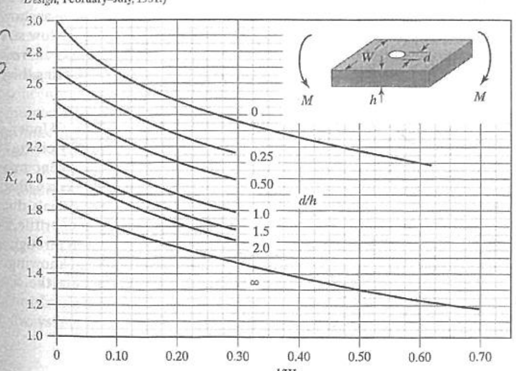

Figure 2-21 Stress concentration factor for a flat bar with a transverse hole in bending. (Spreadsheet Module 2-18.)

(Curves from Peterson, R. E. "Design Factors for Stress Concentration, Parts 1 to 5," *Machine Design*, February–July, 1951.)