ME345 – Modeling and Simulation
Professor Frank Fisher

Problem 1.
Use the 1D finite element code available on the class website to model the problem shown in Figure 1. Neglect the effects of gravity.

The physical parameters of the system are as follows:

Element 1: $A1 = 1\ cm^2$, $E1 = 0.3$ GPa, $\sigma_{ult} = 20$ MPa, $F12 = 50$ N (applied force at intersection of element 1 and 2); polycaprolactone (PCL)
Element 2: $A2 = 0.5\ cm^2$, $E2 = 6$ GPa, $\sigma_{ult} = 57$ MPa, $F23 = 100$ N (applied force at intersection of element 2 and 3); poly(glycolic acid) (PGA)
Element 3: $A3 = 0.25\ cm^2$, $E3 = 12.8$ GPa, $\sigma_{ult} = 52$ MPa, $F3 = 80$ N (applied force at end of bar); transverse cortical bone

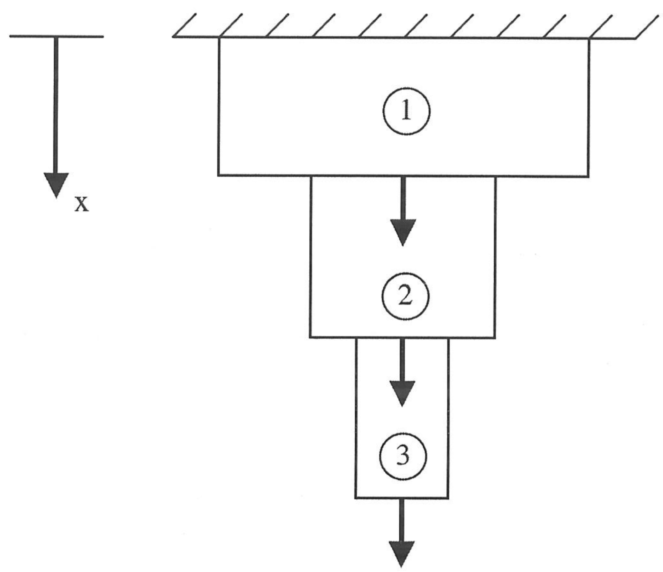
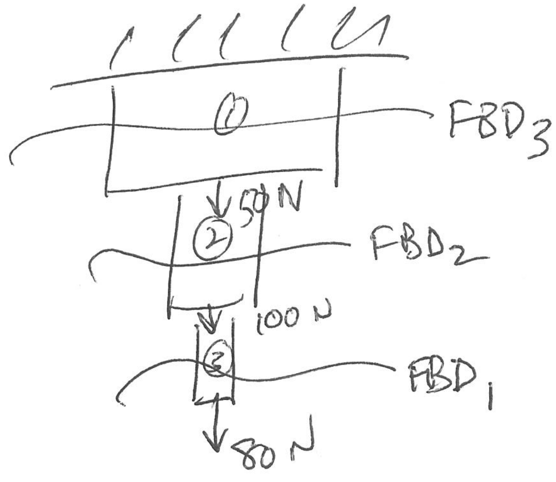


Figure 1. Stepped beam subject to loading. Length of each element is 1 cm.

Submit:
1. A short description of how you edited the Matlab code for the problem. (you can simply cut/paste the top portion of you code here)
2. Plots of stress and displacement versus position x.
3. Hand calculation proving that your Matlab solution is correct.
4. Which material is closest to failing? Why?
5. Short answer to the questions: "What is the difference between the finite element solution and the exact solution for this problem? Why is this How is this different than the icicle problem discussed in class?" (hint: bar/rod elements used here can be called *constant stress* elements.)

Solution 1: old-fashioned way (FBD)



Ele 1

$A_1 = 1\ cm^2$

$E_1 = 0.3\ GPa$

$\sigma_{ult} = 20\ MPa$

Ele 2

$A_2 = 0.5\ cm^2$

$E_2 = 6\ GPa$

$\sigma_{ult} = 57\ MPa$

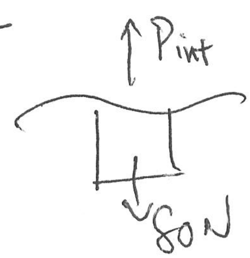
Ele 3

$A_3 = 0.25\ cm^2$

$E_3 = 12.8\ GPa$

$\sigma_{ult} = 52\ MPa$

FBD 1



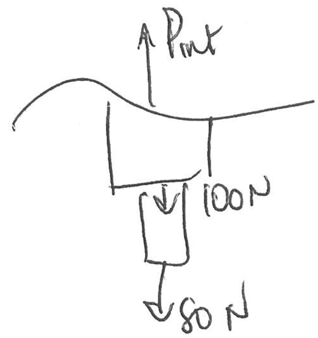
$$\Sigma F_y = 0 \Rightarrow P_{int} = 80\ N$$

$$\sigma_3 = \frac{P_{int}}{A} = \frac{80\ N}{0.25\ cm^2 \times \left(\frac{1\ m}{100\ cm}\right)^2} = \underline{3.2 \times 10^6\ \frac{N}{m^2}}$$

$$SF = \frac{\sigma_{ult}}{\sigma_3} = \frac{52\ MPa}{3.2\ MPa} = \underline{16.25} \text{ (safety factor)}$$

$$\varepsilon_3 = \frac{\sigma_3}{E_3} = \frac{3.2 \times 10^6\ N}{12.8 \times 10^9\ \frac{N}{m^2}} = \underline{2.5 \times 10^{-4} \text{ strain}}$$

FBD2



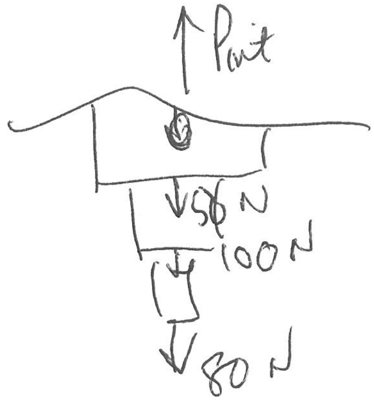
$$\Sigma F_y = 0 \Rightarrow P_{int} = 180\ N$$

$$\sigma_2 = \frac{P_{int}}{A} = \frac{180\ N}{0.5\ cm^2 \times \left(\frac{1}{100}\right)^2} = 3.6 \times 10^6\ N/m^2$$

$$SF = \cancel{3.6 \times 10^6}\ \frac{57\ MPa}{3.6\ MPa} = 15.83$$

$$\varepsilon_2 = \frac{\sigma_2}{E_2} = \frac{3.6 \times 10^6\ N/m^2}{6 \times 10^9\ N/m^2} = \underline{.0006}$$

FBD$_3$



$\Sigma F_y = 0 \Rightarrow P_{int} = 230\text{ N}$

$$\sigma_3 = \frac{230\text{ N}}{1 \times \left(\frac{1}{100}\right)^2} = 2.3 \times 10^6\ \text{N/m}^2$$

$$SF_1 = \frac{20\text{ MPa}}{2.3 \times 10^6} = 8.70$$

$$\varepsilon_1 = \frac{2.3 \times 10^6\ \text{N/m}^2}{0.3 \times 10^9\ \text{N/m}^2} = .0077$$

NOTE: For FEM, we use the external applied forces at each node. When using a FBD approach, we need to calculate the internal forces / stresses in each element. Be sure to understand the difference!

TO CALCULATE THE DISPLACEMENTS, USE THE STRAINS:

Ele 1: $\varepsilon_1 = \frac{\Delta L}{L} \Rightarrow (d_2 - d_1) = (\varepsilon_1)(L_1)$

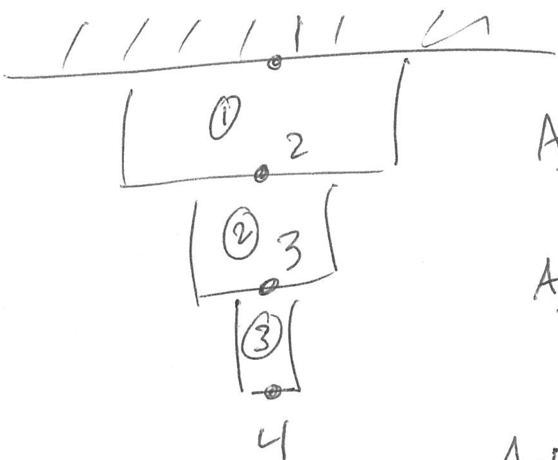
$$d_2 = \varepsilon_1 L_1 + \cancel{d_1}^{0} = (.0077)(1) = .0077\text{ cm}$$

Ele 2: $d_3 = \varepsilon_2 L_2 + d_2 = (.006)(1) + .0077 = .0083\text{ cm}$

Ele 3: $d_4 = \varepsilon_3 L_3 + d_3 = (2.5 \times 10^{-4})(1) + .0083 = .00855\text{ cm}$

ELEMENT WITH THE LOWEST SAFETY FACTOR WILL FAIL FIRST!

Solution 2: FEM by hand



$F_2 = 50\,N \quad F_3 = 100\,N \quad F_4 = 80\,N$

$$\frac{A_1E_1}{L_1} = \frac{(1\,cm^2)(0.3\times10^9\,N/m)}{1\,cm} \times \frac{1\,m}{100\,cm} = \underline{3\times10^6\,N/m}$$

$$\frac{A_2E_2}{L_2} = \frac{(0.5\,cm^2)(6\times10^9\,N/m)}{1\,cm} \times \left(\frac{1\,m}{100\,cm}\right) = \underline{\underline{30\times10^6\,\frac{N}{m}}}$$

$$\frac{A_3E_3}{L_3} = \frac{(0.25\,cm^2)(12.8\times10^9\,N/m)}{1\,cm} \times \left(\frac{1}{100}\right) = \underline{\underline{\overset{32}{\cancel{320}}\times10^6\,\frac{N}{m}}}$$

$$\begin{Bmatrix} F_1 \\ F_2 = 50 \\ F_3 = 100 \\ F_4 = 80 \end{Bmatrix} = 10^6 \begin{bmatrix} 3 & -3 & & \\ -3 & 3+30 & -30 & \\ & -30 & 30+32\cancel{0} & -32\cancel{0} \\ & & -32\cancel{0} & 32\cancel{0} \end{bmatrix} \begin{Bmatrix} d_1 = 0 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

$$\begin{Bmatrix} 50 \\ 100 \\ 80 \end{Bmatrix} = \begin{bmatrix} 33 & -30 & \\ -30 & \cancel{350}\ 62 & \cancel{-320}\ -32 \\ & \cancel{-320}\ -32 & \cancel{320}\ 32 \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

$$\boxed{\begin{Bmatrix} d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} .7667 \\ .8267 \\ .8517 \end{Bmatrix} \times 10^{-4}\ m}$$

Now that have nodal displacements, get strains and stresses...

Ele 1. $\varepsilon_1 = \frac{\Delta L}{L} = \frac{d_2 - \cancel{d_1}^{0}}{L} = \frac{.7667 \times 10^{-4}\ m}{.01\ m} = .0077$

$\sigma_1 = E_1 \varepsilon_1 = (1.3 \times 10^9\ \frac{N}{m^2})(.0077) = \underline{2.3 \times 10^6\ N/m^2}$

Ele 2 $\varepsilon_2 = \frac{\Delta L}{L} = \frac{d_3 - d_2}{L} = \frac{(.8267 - .7667) \times 10^{-4}\ m}{.01\ m} = .0006$

$\sigma_2 = E_2 \varepsilon_2 = (6 \times 10^9\ \frac{N}{m^2})(.0006) = \underline{3.6 \times 10^6\ N/m^2}$

Ele 3 $\varepsilon_3 = \frac{\Delta L}{L} = \frac{d_4 - d_3}{L} = \frac{(.8517 - .8267) \times 10^{-4}}{.01\ m} = 2.5 \times 10^{-4}$

$\sigma_3 = E_3 \varepsilon_3 = (12.8 \times 10^9\ \frac{N}{m^2})(2.5 \times 10^{-4}) = \underline{3.2 \times 10^6\ N/m^2}$

```
% One dimensional finite element program
% Frank Fisher, Stevens Institute of Technology
% Mod/Sim 435, Fall 2007
%
% Edited the standard 1D FEM code to do a specific Stepped-rod problem

clear;
clc;
disp('Franks's code for a specific problem');
disp('From the Figure, node 1 at the top, node 3 at the bottom');
disp('');

numele=3;
numnod=numele+1;


x=[0, 1, 2, 3]*(1/100);                       % x-coordinates of nodes [cm]
node= [1:numele; 2:numele+1];   % node stores the nodes of all elements
area= (1/100)^2*[1, 0.5, 0.25]; % element area [m^2];convert from cm
young=10^9*[0.3, 6, 12.8]; % moduli in Pa

% support conditions, ifix(i)=1 if node i is fixed, else zero
ifix=[1,zeros(1,numele)];
ifix(numnod)=1;          %this will chance the last BC (fixed/free)
ifix(numnod)=0;          %1 = fixed, 0 = free
% note: according to the normal code machinery, this gives fixed/free BCs

% applied forces
force=[0,50, 100, 80];              % applied load [in N]. Note positive down
%
% zero bigk matrix to prepare for assembly
bigk=[zeros(numnod,numnod)];
%
% loop over elements
%
for e=1:numele
    % compute element length
    length=x(node(2,e))-x(node(1,e));
    c=young(e)*area(e)/length;
    % compute element stiffness
    ke=[c,-c;-c,c];

    % assemble ke into bigk
    bigk(node(1,e),node(1,e))=bigk(node(1,e),node(1,e))+ke(1,1);
    bigk(node(1,e),node(2,e))=bigk(node(1,e),node(2,e))+ke(1,2);
    bigk(node(2,e),node(1,e))=bigk(node(2,e),node(1,e))+ke(2,1);
    bigk(node(2,e),node(2,e))=bigk(node(2,e),node(2,e))+ke(2,2);
end

% support conditions (boundary conditions) - normal code
for n=1:numnod
    if (ifix(n) == 1)
        bigk(n,n)=1E+30;
        force(n)=0;
    end
end
```

```
%
disp=force/bigk;         % solve stiffness equations

% compute stresses based on the nodel displacements
for e=1:numele
    % compute element length
    length=x(node(2,e))-x(node(1,e));
    elong=disp(node(2,e))-disp(node(1,e));
    stress(2*e-1)=young(e)*elong/length;
    stress(2*e)=stress(2*e-1);
    xx(2*e-1)=x(node(1,e));
    xx(2*e)=x(node(2,e));
end

% plot displacements and stresses
subplot(211), plot(x,disp,'*')
xlabel('position'), ylabel('displacement');
title('Displacement verses position along stepped rod');
subplot(212)
plot(xx,stress,'*-');
xlabel('position'), ylabel('stress');
title('Stress verses position along stepped rod');
```