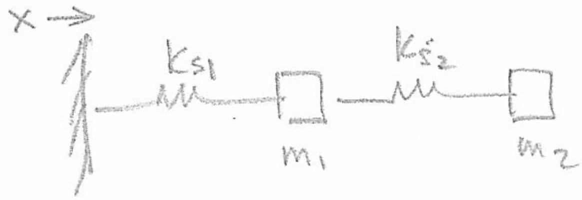
EXAMPLE 7

[redacted]: EIGENVALUE PROBLEM EXAMPLE

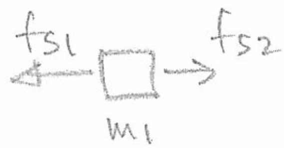


1) CL: $f_{s1} = k_1 x_{s1}$ $\quad f_{s2} = k_2 x_{s2}$

2) GC: $v_{s1} = v_{m1}$ $\quad v_{s2} = v_{m2} - v_{m1}$

"relative velocity"

3) FBD:



$\overset{+}{\rightarrow}$ $\Sigma F_x = m_1 a_{m1}$

$-f_{s1} + f_{s2} = m_1 a_{m1}$

$\overset{+}{\rightarrow}$ $\Sigma F_x = m_2 a_{m2}$

$-f_{s2} = m_2 a_{m2}$

4) State variables: $x_{s1}, x_{s2}, v_{m1}, v_{m2}$ (FOUR SV's here!)

5) $x_{s1}' = v_{s1} = v_{m1}$ ✓

$x_{s2}' = v_{s2} = v_{m2} - v_{m1}$ ✓

$v_{m1}' = a_{m1} = \frac{1}{m_1}(-f_{s1} + f_{s2}) = \frac{1}{m_1}(-k_1 x_{s1} + k_2 x_{s2})$

$v_{m2}' = a_{m2} = \frac{1}{m_2}(-f_{s2}) = \frac{1}{m_2}(-k_2 x_{s2})$ ✓

can write in matrix form:

$$\begin{Bmatrix} x_{s1}' \\ x_{s2}' \\ V_{m1}' \\ V_{m2}' \end{Bmatrix} = \left[\begin{array}{c|c|c|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ \hline -k_1/m_1 & k_2/m_2 & 0 & 0 \\ 0 & -k_2/m_2 & 0 & 0 \end{array}\right] \begin{Bmatrix} x_{s1} \\ x_{s2} \\ V_{m1} \\ V_{m2} \end{Bmatrix}$$

But <u>more useful</u> to write in second order form

$$V_{m1}'' = \frac{d}{dt}(V_{m1}') = \frac{1}{m_1}\left(-k_1 x_{s1}' + k_2 x_{s2}'\right)$$

$$= \frac{1}{m_1}\left[-k_1 V_{s1} + k_2 V_{s2}\right]$$

$$= \frac{1}{m_1}\left[-k_1 V_{m1} + k_2 (V_{m2} - V_{m1})\right] \checkmark$$

$$V_{m2}'' = \frac{d}{dt}(V_{m2}') = \frac{1}{m_2}\left(-k_2 x_{s2}'\right)$$

$$= \frac{1}{m_2}\left(-k_2 V_{s2}\right) = \frac{1}{m_2}\left[-k_2 (V_{m2} - V_{m1})\right] \checkmark$$

$$\begin{Bmatrix} V_{m1}'' \\ V_{m2}'' \end{Bmatrix} = \left[\begin{array}{c|c} \frac{-k_1}{m_1} - \frac{k_2}{m_1} & k_2/m_1 \\ \hline k_2/m_2 & -k_2/m_2 \end{array}\right] \begin{Bmatrix} V_{m1} \\ V_{m2} \end{Bmatrix}$$

we will write this in the form

$$\underset{\sim}{V_m''} = [A] \underset{\sim}{V_m} \quad \text{where } [A] = 2\times 2 \text{ matrix}$$

V_m'', V_m are 2×1.

so we have the problem of

$$\underset{\sim}{V}_m'' = [A]\, \underset{\sim}{V}_m$$

How can we solve this analytically? Let's "guess" a solution... but keep in mind that $\underset{\sim}{V}_m$ is a <u>vector</u>.

a vector of constants

<u>Guess</u> $$\underset{\sim}{V}_m(t) = \underset{\sim}{V}_i\, e^{\lambda t}$$

$$\underset{\sim}{V}_m'(t) = \lambda \underset{\sim}{V}_i\, e^{\lambda t}$$

$$\underset{\sim}{V}_m''(t) = \lambda^2 \underset{\sim}{V}_i\, e^{\lambda t}$$

If this "guess" solution works, then we need:

$$\underset{\sim}{V}_m'' = [A]\, \underset{\sim}{V}_m$$

$$\lambda^2 \underset{\sim}{V}_i\, \cancel{e^{\lambda t}} = [A]\, \underset{\sim}{V}_i\, \cancel{e^{\lambda t}}$$

This is the eigenvalue problem.

$$\left\{ \begin{array}{c} \lambda^2 \underset{\sim}{V}_i = [A]\, \underset{\sim}{V}_i \\ \text{OR} \\ \left([A] - \lambda^2 [I] \right) \underset{\sim}{V}_i = 0 \end{array} \right.$$

$\hookrightarrow$ this form is written in many different ways (i.e. sometimes rather than 'λ' people use 'ω', 'α', etc...)

Although written in different forms, what it means is the same.

★★ Given a matrix $[A]$, can we find scalar quantities λ_i and vectors $\underset{\sim}{v_i}$

that satisfy this "eigenvalue" problem,

where λ_i = eigenvalues
$\underset{\sim}{v_i}$ = eigenvectors (mode shapes)

So $([A] - \lambda^2 [I]) v_i = 0$ ... this can only be true if
1) $\det([A] - \lambda^2 [I]) = 0$, and then
2) if appropriate $\underset{\sim}{v_i}$ are selected.

For our problem,

$$A = \begin{bmatrix} -k_1/m_1 - k_2/m_1 & k_2/m_1 \\ k_2/m_2 & -k_2/m_2 \end{bmatrix}$$

To make easier, let's assume:

$$k_1 = 3,\ k_2 = 2$$
$$m_1 = 1,\ m_2 = 1$$

19

So $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$$[A] - \lambda^2 [I] = \begin{bmatrix} -5-\lambda^2 & 2 \\ 2 & -2-\lambda^2 \end{bmatrix}$$

for what value of λ is this determinant equal to zero?

$$(-5-\lambda^2)(-2-\lambda^2) - 4 = 0$$
$$10 + 5\lambda^2 + 2\lambda^2 + \lambda^4 - 4 = 0$$
$$\lambda^4 + 7\lambda^2 + 6 = 0$$
$$(\lambda^2 + 6)(\lambda^2 + 1) = 0$$

$$\boxed{\lambda_2 = \pm\sqrt{6}\,i \qquad \lambda_1 = \pm i}$$

By convention, we call the smaller value the "first" eigenvalue, etc..

IF we use <u>these</u> eigenvalues, what values of $\underset{\sim}{v_i}$ do we need?

Case 1

$\lambda_1 = \pm i$, $\lambda_1^2 = -1$

$$([A] - \lambda^2 [I])\, \underset{\sim}{v_1} = 0$$

$$\begin{bmatrix} -5-(-1) & 2 \\ 2 & -2-(-1) \end{bmatrix} \begin{Bmatrix} v_{11} \\ v_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{Bmatrix} v_{11} \\ v_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

these are NOT independent! there are infinite number of combinations of v_{11} and v_{12} that will work!

For instance, try $V_{11} = 1$, $V_{12} = 2$
$V_{12} = 2$, $V_{12} = 4$
$V_{12} = 7.6$, $V_{12} = 15.2$
$V_{12} = 1/\sqrt{5}$, $V_{12} = 2/\sqrt{5}$

They all work! But what is important is the direction, not the magnitude, of the vector.

(Easy way) To find these ... set $v_{11} = 1$. then $v_{12} = 2$ to solve the matrix equation.

$$\underset{\sim}{v_1} = \begin{Bmatrix} v_{11} \\ v_{12} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

sometimes this is "normalized" so that magnitude of vector is unit

Case 2

$\lambda_2 = \pm\sqrt{6}\, i \quad , \quad \lambda_2^2 = -6$

$$([A] - \lambda^2 [I])\, \underset{\sim}{V_2} = 0$$

$$\begin{bmatrix} -5-(-6) & 2 \\ 2 & -2-(-6) \end{bmatrix} \begin{Bmatrix} V_{21} \\ V_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} V_{21} \\ V_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

IF $V_{12} = 1$, then $V_{22} = -\frac{1}{2}$.

$$\underset{\sim}{V_2} = \begin{Bmatrix} V_{21} \\ V_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}$$

So we have :

Case 1 $\quad \lambda_1 = \pm i \; , \; \underset{\sim}{V_1} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

Case 2 $\quad \lambda_2 = \pm\sqrt{6}\, i \; , \; \underset{\sim}{V_2} = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}$

these are related to the natural frequencies!

"eigenvectors"
"mode shapes"
"normal modes"

Recall that we started by assuming that

$$\underset{\sim}{V_M}(t) = \underset{\sim}{V_i}\, e^{\lambda t}$$

← this was our "guess"

So we have

$$\underset{\sim}{V}_m(t) = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} e^{\pm i t} + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} e^{\pm \sqrt{6}\, t}$$

Recall from math that we can write these in terms of sin and cos

$$V_m(t) = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} (a_1 \cos t + b_1 \sin t) + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} (a_2 \cos\sqrt{6}\, t + b_2 \sin\sqrt{6}\, t)$$

where (a_1, b_1, a_2, b_2) are constants depending on the initial conditions. (why four constants? Because initially 4 state variables ... 4 1st order state equations)

MODE 1 $\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} f(t)$ →
- Velocities are the same sign
- they are moving "in unison", although mass 2 moving twice as fast

MODE 2 $\begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} f(t)$
- Velocities are opposite signs
- they are moving in opposite directions, with mass 2 speed 50% that of mass 1.

For an arbitrary set of initial conditions, the system response can be written as a SUPERPOSITION of these modes.

For very special cases (very special set of initial conditions), you will only have one mode of behavior.

Let's assume a set of initial conditions and solve for a_1, b_1, a_2, b_2.

NOTE:
- these are NOT accelerations.
- these are NOT damping values.
- these are unknown constants (can also call c_1, c_2, c_3, c_4 / $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ etc.)

ICs: $V_{m1}(t=0) = 0$ $\quad$ $V'_{m1}(t=0) = 3.45$

$V_{m2}(t=0) = 0$ $\quad$ $V'_{m2}(t=0) = .725$

$$\underset{\sim}{V_M} = \begin{Bmatrix}1\\2\end{Bmatrix}(a_1\cos t + b_1\sin t) + \begin{Bmatrix}1\\-1/2\end{Bmatrix}(a_2\cos\sqrt{6}t + b_2\sin\sqrt{6}t)$$

$$\underset{\sim}{V_m'}(t) = \begin{Bmatrix}1\\2\end{Bmatrix}(-a_1\sin t + b_1\cos t) + \begin{Bmatrix}1\\-1/2\end{Bmatrix}\sqrt{6}\,(-a_2\sin\sqrt{6}t + b_2\cos\sqrt{6}t)$$

at $t=0$, $\sin 0 = 0$ $\Rightarrow$ plug this in
$\cos 0 = 1$ (check if not sure)

$V_m(t=0)$ ICs:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}(a_1) + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} a_2$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1/2 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \Rightarrow \begin{matrix} a_1 = 0 \\ a_2 = 0 \end{matrix}$$

$\dot{V}_m(t=0)$ ICs:

$$\begin{Bmatrix} 3.45 \\ .775 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}(b_1) + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix}\sqrt{6}\,(b_2)$$

$$\begin{Bmatrix} 3.45 \\ .775 \end{Bmatrix} = \begin{bmatrix} 1 & \sqrt{6} \\ 2 & -\sqrt{6}/2 \end{bmatrix}\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \Rightarrow \begin{matrix} b_1 = 1 \\ b_2 = 1 \end{matrix}$$

PLUG THESE INTO THE GENERAL SOLUTION

$$\underset{\sim}{V_m}(t) = \begin{Bmatrix} V_{m1}(t) \\ V_{m2}(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}\sin t + \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix}\sin\sqrt{6}$$