Estimation of the long memory parameter in stochastic volatility models by quadratic variations

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Abstract

We consider a stochastic volatility model where the volatility process is a fractional Brownian motion. We estimate the memory parameter of the volatility from discrete observations of the price process. We use criteria based on Malliavin calculus in order to characterize the asymptotic normality of the estimators.

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1 Introduction

We consider a stochastic volatility model where the volatility is driven by a fractional Brownian motion (fBM). Such a model is motivated by recent work that shows the long

*Associate member of the team Samm, Université de Panthéon-Sorbonne Paris 1 range dependence of the volatility process (Cont, 2005; Casas and Gao, 2008; Mariani et al., 2009; Chronopoulou and Viens, 2010).

We assume as given a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, endowed with a complete filtration $\mathscr{F} = \{\mathscr{F}_t\}_{t\geq 0}$ (see Protter, 2005, page 3). On this space we observe the process:

$$X_t = \int_0^t B_s^H dW_s, \tag{1}$$

where the processes $\{W_t\}_{t\in(0,\infty)}$ a standard Brownian motion and $\{B_t^H\}_{t\in(0,\infty)}$ a fractional Brownian motion are independent and are both adapted with respect to the filtration $\mathscr{F}$. Recall that the fractional Brownian motion is a centered Gaussian process with covariance, for every $s, t \geq 0$,

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

It may also be characterized as the only Gaussian self-similar process with stationary increments. We will denote by $\mathcal{F}_t^W$ and by $\mathcal{F}_t^{B^H}$ the σ-algebras generated by W and B^H respectively.

We consider the problem of estimating the Hurst parameter H given discrete observations of the process X: $(X_0, X_{t_1}, X_{t_2}, \ldots, X_{t_N})$ for a discrete partition $\pi = (t_0 = 0 < t_1 < \cdots < t_N = 1)$. The end point is taken to be $t_N = 1$ for convenience.

This model is a particular case of a stochastic volatility model. We chose to work with the fractional Brownian motion due to its well documented long memory behavior. We mention related work Gloter and Hoffmann (2004) where a more general problem is presented: given discrete time observations of the process $X_t = \int_0^t \phi(\theta, B_s^H) dW_s$ estimate the parameter θ. In the cited work both the functional form of $\phi(\cdot,\cdot)$ and the Hurst parameter H are assumed to be known. Under these assumptions the estimator θ_n (Gloter and Hoffmann, 2004, formulas (13)-(15)) is proven consistent and asymptotically normal (θ in fact depends explicitly on H). In more recent work (Chronopoulou and Viens, 2010) the authors show that knowing H is crucial for determining the optimal value of the parameter θ. This motivated us to consider a simple model first where the properties of the estimator may be be properly analyzed.

Our estimator for the Hurst parameter will be constructed using the quadratic variations of the process X. The use of the quadratic variations for estimating the self-similarity index of a self-similar process is standard and it has been widely used in the literature. We refer, among others, to Bardet and Tudor (2010), Coeurjolly (2001), Istas and Lang (1997) or Tudor and Viens (2009). Indeed, it is well-known that, if we observe a self-similar process (for example the fractional Brownian motion) at discrete times $\frac{i}{N}$, with $i = 1, 2, ..., N$ then $\hat{H}_N := -\frac{\log S_N}{2 \log N}$ is a consistent and asymptotically normal estimator for H. In the formula above $S_N = \frac{1}{N}\sum_{i=0}^{N-1}\left(B^H_{\frac{i+1}{N}} - B^H_{\frac{i+1}{N}}\right)^2$, and note that the process needs to be observed directly in order to construct the estimator.

Our context is different. First, the process appears in the volatility part and the volatility in general is not directly observable. Second, although the volatility process B^H is self-similar, the observed process X is not self-similar anymore. However, the quadratic variation process of X will play an important role in our estimation. Specifically, if we denote by

$$V_N(t) = \sum_{i=1}^{N}\left(X_{t_i} - X_{t_{i-1}}\right)^2 \tag{2}$$

the quadratic variation, with

$$t_0 = 0 < t_1 < \dots t_N = t \tag{3}$$

a partition of the interval $[0, t]$ then, by a classical result in martingale theory,

$$V_N(1) = \sum_{i=1}^{N}\left(X_{t_i} - X_{t_{i-1}}\right)^2 \overset{N\to\infty}{\longrightarrow} \int_0^1 (B^H_s)^2 ds,$$

when $\|\pi\| = \max_{i\in\{1,2,\dots,N\}}(t_i - t_{i-1}) \to 0$ and the convergence holds (at least) in probability.

We consider,

$$\theta = I\!E\left[\int_0^1 (B^H_s)^2 ds\right] = \int_0^1 s^{2H} ds = \frac{1}{2H+1}.$$

The natural estimator for θ clearly is $V_N(1)$. The idea of the present work is to analyze the properties of this estimator and then to obtain an estimator for H.

The article is structure in the following way. Section 2 presents the tools we use in our technical analysis. Section 3 analyzes the properties of the straight quadratic variation process. We demonstrate that this estimator, although unbiased, it is not L^2 consistent and in fact we find the limiting value for its variance. Therefore, an estimator based on the straight quadratic variance will not get better as the number of observations increase. In section 3.2 we present another estimator (conditional Quadratic Variance). We show that this estimator is strong consistent. Finally, section 4 presents the asymptotic normality properties of this estimator.

2 Preliminaries on Malliavin calculus and multiple stochastic integrals

Let $(W_t)_{t\in[0,1]}$ be a classical Wiener process on a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0,1]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of f with respect to W. We mention the basic reference Nualart (2006) for background on the notions we use. Let $\mathcal{S}_n$ be the space of elementary functions with n variables:

$$\mathcal{S}_n = \left\{ f \in L^2([0,1]^n) : f = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} 1_{A_{i_1}\times\ldots\times A_{i_n}},\ A_i \in \mathcal{B}([0,1]) \right\}$$

where the coefficients satisfy $c_{i_1,\ldots,i_m} = 0$ if any two indices i_k and i_l are equal or any of the sets are empty. For such a step function $f \in \mathcal{S}_n$ we define

$$I_n(f) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots i_n} W(A_{i_1}) \ldots W(A_{i_n})$$

where we use the notation $W(A) = \int_0^1 1_A(s) dW_s$. If we denote $\tilde{f}$ the symmetrization of f:

$$\tilde{f}(x_1, \ldots, x_x) = \frac{1}{n!} \sum_{\sigma\in S_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

we also have that $I_n(f) = I_n\big(\tilde{f}\big)$. Furthermore, the functional I_n constructed above from $\mathcal{S}_n$ to $L^2(\Omega)$ is an isometry on $\mathcal{S}_n$, i.e.

$$I\!E\left[I_m(f)I_n(g)\right] = \begin{cases} n!\langle \tilde{f}, \tilde{g}\rangle_{L^2([0,1]^n)}, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} \tag{4}$$

The set $\mathcal{S}_n$ is dense in $L^2([0,1]^n)$ for every $n \geq 1$ and thus the mapping I_n may be extended to an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension as well.

We will need the general formula for calculating products of Wiener chaos integrals of any orders m, n for any symmetric integrands $f \in L^2([0,1]^{\otimes m})$ and $g \in L^2([0,1]^{\otimes n})$; this formula is

$$I_m(f)I_n(g) = \sum_{r=0}^{m\wedge n} r! \binom{m}{r} \binom{n}{r} I_{m+n-2r}(f \otimes_r g) \tag{5}$$

where the contraction $f \otimes_r g$ is defined by

$$\begin{aligned} &(f \otimes_\ell g)(s_1, \ldots, s_{m-\ell}, t_1, \ldots, t_{n-\ell}) \\ &= \int_{[0,T]^{m+n-2\ell}} f(s_1, \ldots, s_{m-\ell}, u_1, \ldots, u_\ell) g(t_1, \ldots, t_{n-\ell}, u_1, \ldots, u_\ell) du_1 \ldots du_\ell. \end{aligned} \tag{6}$$

Note that the contraction $(f \otimes_\ell g)$ is an element of $L^2([0,1]^{m+n-2\ell})$ but it is not necessary symmetric. We will denote by $(f \tilde{\otimes}_\ell g)$ its symmetrization.

We recall that any square integrable random variable measurable with respect to the σ-algebra generated by W may be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n \geq 0} I_n(f_n) \tag{7}$$

where $f_n \in L^2([0,1]^n)$ are (uniquely determined) symmetric functions and $I_0(f_0) = I\!E\,[F]$.

We denote by D the Malliavin derivative operator that acts on smooth functionals of the form $F = g(W(\varphi_1), \ldots, W(\varphi_n))$

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i,$$

here g is a smooth function with compact support and $\varphi_i \in L^2([0,1])$ for $i = 1, .., n$.

The operator D can be extended to the closure $\mathbb{D}^{p,2}$ of smooth functionals with respect to the norm

$$\|F\|_{p,2}^2 = I\!E F^2 + \sum_{i=1}^{p} I\!E \|D^i F\|^2_{L^2[0,1]^i}$$

where the i-th Malliavin derivative $D^{(i)}$ is defined iteratively.

If $f \in L^2([0,1]^n)$ we will use the following rule to differentiate multiple integrals in the Malliavin sense

$$D_t I_n(f) = n I_{n-1}(f(\cdot, t)), \quad t \in [0,1],$$

and the rule may be extended easily to any square integrable random variable F using the representation (7).

The adjoint of D is denoted by δ and is called the divergence operator (Skorohod integral in the white noise case). Its domain ($Dom(\delta)$) coincides with the class of stochastic processes $u \in L^2(\Omega \times [0,1])$ such that

$$\left| I\!E \langle DF, u \rangle_{L^2([0,1])} \right| \leq c \|F\|_2$$

for all $F \in \mathbb{D}^{1,2}$ and $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the duality relationship

$$I\!E(F\delta(u)) = I\!E \langle DF, u \rangle_{L^2([0,1])}. \tag{8}$$

For adapted integrands, the divergence integral coincides to the classical Itô integral. Also we will need in the paper the integration by parts formula

$$F\delta(u) = \delta(Fu) + \langle DF, u \rangle \tag{9}$$

whenever $F \in \mathbb{D}^{1,2}$, $u \in Dom(\delta)$ and $I\!\!E F^2 \int\limits_0^1 u_s^2 ds < \infty$.

3 On the consistency of the quadratic variation

Let us consider the process X given by (1) with B^Hindependent of W. The quadratic variation of X is defined by the formula (2). As we mentioned in the introduction, $I\!\!E[V_N(1)] = \theta = \frac{1}{2H} + 1$, for all N.

$$\begin{aligned} I\!\!E\left[\sum_{i=1}^{N}\left(X_{t_i} - X_{t_{i-1}}\right)^2\right] &= \sum_{i=1}^{N} I\!\!E\left[\int_{t_{i-1}}^{t_i} B_s^H dW_s\right]^2 = \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} I\!\!E\left[B_s^H\right]^2 ds \\ &= \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} s^{2H} ds = \sum_{i=1}^{N} \frac{t_i^{2H+1}}{2H+1} - \frac{t_{i-1}^{2H+1}}{2H+1} = \frac{1}{2H+1}. \end{aligned}$$

Therefore, the estimator $V_N(1)$ is unbiased for the parameter $\theta = \frac{1}{2H+1}$. In the next section we discuss the consistency of this estimator. We look at the convergence of $V_N(1)$ to the parameter θ in $L^2(\Omega)$ and almost surely. The almost sure convergence is very useful in order to estimate the Hurst parameter H itself.

3.1 Consistency

There are several notions of consistency that we may consider.

Definition 1 *Suppose have an estimator T_N estimating a certain parameter θ. The estimator is called **weakly consistent** if $T_N \to \theta$ in probability. The estimator is called **strongly consistent** if $T_N \to \theta$ almost surely. The estimator is called L^2 **consistent** if $I\!\!E(T_N - \theta)^2$ converges to zero. All the limits are with respect to $N \to \infty$.*

We will first discuss the $L^2(\Omega)$ consistency of the estimator $V_N(1)$ given by (2). Please note that due to the unbiasedness of the estimator for a fixed N the expression $I\!\!E(T_N - \theta)^2$ is also the variance of the estimator.

$$I\!\!E\left[(V_N(1) - \theta)^2\right] = I\!\!E[V_N(1)^2] - \theta^2$$

and,

$$
\begin{aligned}
I\!E[V_N(1)^2] &= I\!E\left[\sum_{i,j=1}^{N}\left(X_{t_i}-X_{t_{i-1}}\right)^2\left(X_{t_j}-X_{t_{j-1}}\right)^2\right]\\
&= \sum_{i=1}^{N} I\!E\left[\left(X_{t_i}-X_{t_{i-1}}\right)^4\right]\\
&\quad +2\sum_{i<j}^{N} I\!E\left[\left(X_{t_i}-X_{t_{i-1}}\right)^2\left(X_{t_j}-X_{t_{j-1}}\right)^2\right] \qquad (10)
\end{aligned}
$$

To proceed with the calculation we need several results.

Lemma 1 *Let $a<b$ and $s<t$. Then*

$$
I\!E\left(X_t-X_s\right)^2\left(X_b-X_a\right)^2 = I\!E\int_s^t (B_u^H)^2du\int_a^b (B_u^H)^2du + 2I\!E\left(\int_{[s,t]\cap[a,b]}(B_u^H)^2du\right)^2 \qquad (11)
$$

Proof: Suppose first that $[a,b]=[s,t]$. Since the conditional law of X_t-X_s given $\mathcal{F}^{B^H}$ is

$$
\left(\int_s^t (B_s^H)^2ds\right)^{\frac{1}{2}} Z
$$

with Z a standard normal random variable independent by B^H we get

$$
I\!E\left(X_t-X_s\right)^4 = I\!E\left[I\!E(X_t-X_s)^4\,\middle|\,\mathcal{F}^{B^H}\right] = 3I\!E\int_s^t (B_u^H)^2du.
$$

In the general case, we use the techniques of the Malliavin calculus. The Malliavin derivatives, throughout the paper, are defined with respect to W. From the independence of B^H and W, it follows that $Df(B_t^H)=0$ for every t and every function f. We can write, using (8)

$$
\begin{aligned}
& \mathbb{E}\left(X_t - X_s\right)^2 \left(X_b - X_a\right)^2 \\
= & \ \mathbb{E}\int_s^t B_u^H dW_u \int_s^t B_u^H dW_u \left(\int_a^b B_x^H dW_x\right)^2 \\
= & \ \mathbb{E}\int_s^t du B_u^H D_u \left[\int_s^t B_u^H dW_u \left(\int_a^b B_x^H dW_x\right)^2\right] \\
= & \ \mathbb{E}\int_s^t du B_u^H \left[1_{[s,t]}(u) B_u^H \left(\int_a^b B_x^H dW_x\right)^2 + 2\int_s^t B_v^H dW_v \int_a^b B_x^H dW_x 1_{[a,b]}(u) B_u^H\right] \\
= & \ \mathbb{E}\int_s^t du (B_u^H)^2 \left(\int_a^b B_x^H dW_x\right)^2 + 2\int_{[s,t]\cap[a,b]} du (B_u^H)^2 \int_s^t B_v^H dW_v \int_a^b B_x^H dW_x \\
= & \ \mathbb{E}\int_s^t du \int_a^b dx B_x^H D_x \left[(B_u^H)^2 \int_a^b B_x^H dW_x\right] \\
& +2\mathbb{E}\int_{[s,t]\cap[a,b]} du \int_s^t dv B_v^H (B_u^H)^2 1_{[a,b]}(v) B_v^H \\
= & \ \mathbb{E}\int_s^t (B_u^H)^2 du \int_a^b (B_u^H)^2 du + 2\mathbb{E}\left(\int_{[s,t]\cap[a,b]} (B_u^H)^2 du\right)^2.
\end{aligned}
$$

■

Lemma 2 *Let $u, v \geq 0$. Then*

$$
E\left[(B_u^H)^2 (B_v^H)^2\right] = 2R_H(u,v) + u^{2H} v^{2H}. \tag{12}
$$

Proof: We use Gaussian regression. We recall that, if (G_1, G_2) is a Gaussian vector then

$$
G_2 = \frac{Cov(G_1,G_2)}{Var(G_1)} G_1 + \sqrt{Var(G_2) - \frac{Cov(G_1,G_2)^2}{Var(G_1)}} Z \tag{13}
$$

where Z is a standard normal random variable independent by G_1. In our case applying (13) to the Gaussian couple (B_u^H, B_v^H) we get

$$
B_u^H = \frac{R_H(u,v)}{v^{2H}} B_v^H + \sqrt{u^{2H} - \frac{R_H(u,v)^2}{v^{2H}}} Z
$$

with Z a standard normal random variable independent by B_v^H. Then

$$
B_u^H B_v^H = \frac{R_H(u,v)}{v^{2H}} (B_v^H)^2 + \sqrt{u^{2H} - \frac{R_H(u,v)^2}{v^{2H}}} Z B_v^H
$$

and

$$\begin{aligned}
\mathbb{E}\left(B_u^H B_v^H\right)^2 &= \frac{R_H(u,v)^2}{v^{4H}}\mathbb{E}(B_v^H)^4 + \left(u^{2H} - \frac{R_H(u,v)^2}{v^{2H}}\right)\mathbb{E}Z^2(B_v^H)^2 \\
&= 3\frac{R_H(u,v)^2}{v^{2H}} + \left(u^{2H} - \frac{R_H(u,v)^2}{v^{2H}}\right)v^{2H} \\
&= 2R_H(u,v)^2 + u^{2H}v^{2H}.
\end{aligned}$$

■

We are now able to compute $\mathbb{E}V_N(1)^2$. Using formula (11),

$$\begin{aligned}
\mathbb{E}V_N(1)^2 &= \sum_{i=0}^{N-1}\mathbb{E}\left(X_{t_{i+1}} - X_{t_i}\right)^4 + 2\sum_{i>j}\mathbb{E}\left(X_{t_{i+1}} - X_{t_i}\right)^2\left(X_{t_{j+1}} - X_{t_j}\right)^2 \\
&= 3\sum_{i=0}^{N-1}\mathbb{E}\left(\int_{t_i}^{t_{i+1}}(B_u^H)^2du\right)^2 \\
&\quad +2\sum_{i>j}\mathbb{E}\int_{t_i}^{t_{i+1}}(B_u^H)^2du\int_{t_j}^{t_{j+1}}(B_u^H)^2du. \qquad (14)
\end{aligned}$$

Next, we need the following lemma which shows that the diagonal term above converges to zero.

Lemma 3 *Define*

$$A_N := \sum_{i=0}^{N-1}\mathbb{E}\left(\int_{t_i}^{t_{i+1}}(B_u^H)^2du\right)^2.$$

Then $A_N \to 0$ as $N \to \infty$.

Proof: Using Lemma 2 we can write

$$\begin{aligned}
A_N &= \mathbb{E}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}(B_u^H)^2(B_v^H)^2dvdu \\
&= \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\left(2R_H(u,v)^2 + u^{2H}v^{2H}\right)dvdu
\end{aligned}$$

Consider $t_i = \frac{i}{N}$ for $i = 0, .., N$. Note first that

$$\begin{aligned}
\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}u^{2H}v^{2H}dvdu &= \frac{1}{(2H+1)^2}\frac{1}{N^{4H+2}}\sum_{i=0}^{N-1}\left((i+1)^{2H+1} - i^{2H+1}\right)^2 \\
&= \frac{1}{(2H+1)^2}\frac{1}{N^{4H+2}}\sum_{i=0}^{N-1}i^{4H+2}f(\frac{1}{i})^2
\end{aligned}$$

with $f(x) = (1+x)^{2H+1} - 1$. The function f behaves as $(2H+1)x$ when x goes to zero. Therefore

$$\frac{1}{(2H+1)^2} \frac{1}{N^{4H+2}} \sum_{i=0}^{N-1} i^{4H+2} f(\frac{1}{i})^2$$

behaves, as $N \to \infty$ as

$$\frac{1}{N^{4H+2}} \sum_{i=0}^{N-1} i^{4H} \sim \frac{1}{4H+1} \frac{1}{N}$$

and this goes to zero as $N \to \infty$. Now, let us estimate the quantity

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} R_H(u,v)^2.$$

We can write

$$\begin{aligned} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} R_H(u,v)^2 dvdu &= \frac{1}{4} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \left(u^{2H} + v^{2H} - |u-v|^{2H}\right)^2 dudv \\ &\leq 2 \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \left(u^{4H} + v^{4H} + |u-v|^{4H}\right) dvdu. \end{aligned}$$

The three above summands converge to zero s $N \to \infty$. Indeed

$$\begin{aligned} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} u^{4H} dvdu &= \frac{1}{4H+1} \sum_{i=0}^{N-1} (t_{i+1} - t_i)(t_{i+1}^{4H+1} - t_i^{4H+1}) \\ &= \frac{1}{4H+1} \frac{1}{N^{4H+2}} \sum_{i=0}^{N-1} \left((i+1)^{4H+1} - i^{4H+1}\right) \end{aligned}$$

and this behaves as $\frac{1}{N} \to 0$ as $N \to \infty$. It remains to check that

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} |u-v|^{4H} dvdu$$

converges to zero as $N \to \infty$. This is obvious since

$$\begin{aligned}
\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} |u-v|^{4H} dv du &= 2\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} (u-v)^{4H} dv du \\
&= \frac{1}{4H+1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} (u-t_i)^{4H+1} du \\
&= \frac{1}{(4H+1)(4H+2)} \sum_{i=0}^{N-1} (t_{i+1}-t_i)^{4H+2} \\
&= \frac{1}{(4H+1)(4H+2)} \frac{1}{N^{4H+1}} \to_{N\to\infty} 0.
\end{aligned}$$

■

We restate the last result obtained.

Proposition 1 *As $N \to \infty$,*

$$I\!\!E V_N(1)^2 \to \int_0^1 \int_0^1 I\!\!E (B_u^H)^2 (B_v^H)^2 dv du = \int_0^1 \int_0^1 (2R_H(u,v)^2 + u^{2H} v^{2H}) du dv.$$

Proof: This is a consequence of relation (14), Lemma 2 and Lemma 3. ■

We will prove now that the estimator $V_N(1)$ has non-trivial quadratic error, i.e. it is not L^2 consistent.

Theorem 1 *The estimator $V_N(1)$ has non-trivial limiting quadratic error given by*

$$\lim_{N\to\infty} I\!\!E \left(V_N(1)^2 - \frac{1}{2H+1} \right)^2 = 2 \int_0^1 \int_0^1 R_H(u,v)^2 du dv.$$

Proof: The result follows from relation (10) and Proposition 1. ■.

Remark 1 *The fact that the estimator $V_N(1)$ does not convergence in $L^2(\Omega)$ to $\theta = \frac{1}{2H+1}$ is not really surprising. In fact, we may write*

$$V_N(1) - \frac{1}{2H+1} = V_N(1) - \int_0^1 (B_s^H)^2 ds + \int_0^1 (B_s^H)^2 ds - \frac{1}{2H+1} \tag{15}$$

The first part $V_N(1) - \int_0^1 (B_s^H)^2 ds$ converges to zero as $N \to \infty$ (at least in probability) while the second part does not depends on N and does not vanish. Moreover the difference

$\int_0^1 (B_s^H)^2 ds - \frac{1}{2H+1}$ can be written as, by Itô's formula for the fBm (see Nualart (2006), Chapter 5)

$$\int_0^1 (B_s^H)^2 ds - \frac{1}{2H+1} = \int_0^1 ds \left(\int_0^s B_v^H dB_v^H + s^{2H} \right) - \frac{1}{2H+1} = \int_0^1 ds \int_0^s B_v^H dB_v^H.$$

Here dB^H denotes the divergence integral with respect to B^H (see Nualart (2006) for details). Therefore, the limiting variation of $V_N(1)$ is in fact the second moment of the random variable $\int_0^1 ds \int_0^s B_v^H dB_v^H$. It may also be noticed that the almost sure convergence of $V_N(1)$ to θ cannot hold.

3.2 Conditional consistency

Let us go back to Remark 1, more precisely to relation (15). Consider $\mathcal{F}_t^W$ be the filtration generated by the Wiener process W and let us replace the estimator with the conditional expectation given $\mathcal{F}_t^W$ in (15). From the independence of W and B^H, the quantity $\int_0^1 (B_s^H)^2 ds - \frac{1}{2H+1}$ vanishes conditional on $\mathcal{F}_1^W$ or with respect to any other σ algebra depending only on W.

In fact it is possible to explicitly compute the conditional expectation $\mathbb{E}\left[V_N(1) \mid \mathcal{F}_1^W\right]$.

Proposition 2 *We have, for every $N \geq 1$ and for any partition Δ_N of the interval $[0,1]$ of the form*

$$\mathbb{E}\left[V_N(1) \mid \mathcal{F}_1^W\right] = \frac{1}{2H+1} + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} R_H(u,v) dW_u dW_v. \tag{16}$$

Proof: Note that $\mathbb{E}\left[V_N(1) \mid \mathcal{F}_1^W\right] = \sum_{i=1}^{N-1} \mathbb{E}\left[\left(X_{t_{i+1}} - X_{t_i}\right)^2 \mid \mathcal{F}_1^W\right]$. Thus we only need to compute the quantity

$$\mathbb{E}\left[(X_t - X_s)^2 \mid \mathcal{F}_1^W\right]$$

for any $s,t \geq 0$, $s \leq t$. Note that we may write $X_t - X_s$ as an $L^2(\Omega)$–limit of the process $S_{s,t}(M)$, where

$$S_{s,t}(M) := \sum_{j=0}^{M-1} B_{s_j}^H \left(W_{s_{j+1}} - W_{s_j}\right).$$

and $s = s_0 < s_1 < \ldots < s_M = t$ denotes a partition of the interval $[s,t]$. We first calculate

$$
\begin{aligned}
I\!\!E\left[S_{s,t}(M)^2 \mid \mathcal{F}_1^W\right] &= I\!\!E\left[\sum_{i,j=0}^{M-1} B_{s_i}^H B_{s_j}^H \left(W_{s_{i+1}} - W_{s_i}\right)\left(W_{s_{j+1}} - W_{s_j}\right) \middle| \mathcal{F}_1^W\right] \\
&= \sum_{i,j=0}^{M-1} R_H(s_i, s_j)\left(W_{s_{i+1}} - W_{s_i}\right)\left(W_{s_{j+1}} - W_{s_j}\right) \\
&= \sum_{i=0}^{M-1} R_H(s_i, s_i)\left(W_{s_{i+1}} - W_{s_i}\right)^2 \\
&\quad + \sum_{i,j=0; i\neq j}^{M-1} R_H(s_i, s_j)\left(W_{s_{i+1}} - W_{s_i}\right)\left(W_{s_{j+1}} - W_{s_j}\right).
\end{aligned}
$$

The first summand above equals

$$\sum_{i=0}^{M-1} s_i^{2H}\left(W_{s_{i+1}} - W_{s_i}\right)^2$$

and by classical techniques of the martingales theory, it converges in $L^1(\Omega)$ to

$$\int_s^t u^{2H} ds = \frac{1}{2H+1}\left(t^{2H+1} - s^{2H+1}\right).$$

The non-diagonal term converges in $L^2(\Omega)$ to

$$\int_s^t \int_s^t R_H(u,v) dW_u dW_v$$

by the construction of the multiple integral with respect to the Wiener process W as introduced in Section 2 (double integral in this case). $\blacksquare$

We next investigate the convergence in L^2 of the random variable $I\!\!E\left(V_N(1)/\mathcal{F}_1^W\right)$. Since we shall use the result afterwards we would like to calculate not only the limit but also the order of convergence.

Let us denote the L^2 norm

$$a_N^2 := I\!\!E\left[\left(I\!\!E\left[V_N(1) \mid \mathcal{F}_1^W\right] - \frac{1}{2H+1}\right)^2\right] \tag{17}$$

Using the previous result (Proposition 2) and the isometry formula for multiple stochastic integrals, it holds that,

$$\begin{aligned}
a_N^2 &:= \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{\frac{i+1}{N}} R_H(u,v)^2 du dv \\
&= \frac{1}{4} \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \left(u^{2H} + v^{2H} - |u-v|^{2H}\right)^2 du dv \\
&= \frac{1}{4} \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \left(2u^{4H} + 2u^{2H}v^{2H} - 4u^{2H}|u-v|^{2H} + |u-v|^{4H}\right) du dv. \quad (18)
\end{aligned}$$

where we used the symmetry of the integrals in u and v.

It is an easy exercise to estimate the behavior as $N \to \infty$ of the four resulting sums (we also recall the similar calculations in Lemma 3). The first one may be handled as follows.

$$\begin{aligned}
\sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{\frac{i+1}{N}} u^{4H} du dv &= \frac{1}{4H+1} \frac{1}{N^{4H+2}} \sum_{i=0}^{N-1} \left((i+1)^{4H+1} - i^{4H+1}\right) \\
&= \frac{1}{4H+1} \frac{1}{N} = \mathcal{O}(N^{-1})
\end{aligned}$$

Concerning the second summand in (3)

$$\sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{\frac{i+1}{N}} u^{2H}v^{2H} du dv = \frac{1}{(2H+1)^2} \frac{1}{N^{4H+2}} \sum_{i=0}^{N-1} ((i+1)^{2H+1} - i^{2H+1})^2 = \mathcal{O}(N^{-1})$$

and for the third summand in (3) we can write

$$\begin{aligned}
&\sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{\frac{i+1}{N}} u^{2H}|u-v|^{2H} du dv \\
&= \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{u} u^{2H}(u-v)^{2H} dv du + \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{u}^{\frac{i+1}{N}} u^{2H}(v-u)^{2H} dv du \\
&= \frac{1}{2H+1} \sum_{i=0}^{N-1} \left[\int_{\frac{i}{N}}^{\frac{i+1}{N}} u^{2H} \left(u - \frac{i}{N}\right)^{2H+1} du + \int_{\frac{i}{N}}^{\frac{i+1}{N}} u^{2H} \left(\frac{i+1}{N} - u\right)^{2H+1} du\right].
\end{aligned}$$

Changing to the variables $z = N\left(u - \frac{i}{N}\right)$ in the first integral and $z = N\left(\frac{i+1}{N} - u\right)$ in the second integral we get that the above sum is equal to

$$\frac{1}{2H+1} \frac{1}{N^{4H+2}} \sum_{i=0}^{N-1} \left[\int_0^1 z^{2H+1}(z+i)^{2H} dz + \int_0^1 z^{2H+1}(i+1-z)^{2H} dz\right]$$

and this is of order of $\mathcal{O}(N^{-2H-1})$.

A similar calculation shows,

$$\sum_{i=0}^{N-1}\int_{\frac{i}{N}}^{\frac{i+1}{N}}\int_{\frac{i}{N}}^{\frac{i+1}{N}}|u-v|^{4H}dudv \quad = \quad 2\sum_{i=0}^{N-1}\int_{\frac{i}{N}}^{\frac{i+1}{N}}\int_{\frac{i}{N}}^{u}(u-v)^{4H}dudv = \mathcal{O}(N^{-4H-1}).$$

Note that regardless of H, $a_n^2 \to 0$ at the order $\mathcal{O}(N^{-1})$. We restate the result as a proposition to be used later.

Proposition 3 *Let a_N be given by*

$$a_N^2 := I\!\!E\left[\left(I\!\!E\left[V_N(1)\mid \mathcal{F}_1^W\right]-\frac{1}{2H+1}\right)^2\right].$$

Then, as $N\to\infty$,

$$a_N = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

We obtain the L^2 consistency and the strong consistency as immediate consequences.

Corollary 1 *Let $\mathcal{F}^W$ be the sigma-algebra generated by $(W_t)_{t\in[0,1]}$. Then*

$$I\!\!E\left[V_N(1)\mid \mathcal{F}_1^W\right] \overset{N\to\infty}{\longrightarrow} \theta = \frac{1}{2H+1}.$$

in $L^2(\Omega)$. If V_N is constructed via a refining partition (that if, $\Delta_N\subset\Delta_M$ if $N>M$) then the above convergence holds almost surely.

Proof: We have already shown the L^2 convergence above.
Concerning the almost sure convergence, we may proceed as in (Protter, 2005, Theorem 28, Chapter I), and show that $V_N(1)$ converges almost surely to $\int_0^1 (B_s^H)^2 ds$ if we work with refining partitions. This may be proved using the dominated convergence theorem for conditional expectations, that $I\!\!E\left[V_N(1)\mid \mathcal{F}_1^W\right]$ converges as $N\to\infty$ almost surely, to $I\!\!E\left(\int_0^1 (B_s^H)^2 ds \mid \mathcal{F}^W\right) = \frac{1}{2H+1}$. ■

From the above result we immediately obtain

Proposition 4 *Define for every $N\geq 1$,*

$$\hat{H}_N := \frac{1}{2I\!\!E\left[V_N(1)\mid \mathcal{F}_1^W\right]} - \frac{1}{2}.$$

Assume that V_N is constructed via a refining partition. Then $\hat{H}_N$ converge almost surely to H as $N\to\infty$.

Remark 2 *The estimator $I\!E\left[V_N(1) \mid \mathcal{F}_1^W\right]$ has good properties. Its inconvenient is that in principle it cannot be directly from the observations of the process X. Nevertheless, Proposition (16) shows that it has a rather simple form which could allows its numerical simulation at least in particular cases. This will be the object of a further study.*

4 Conditional central limit theorem

Let us discuss the asymptotic behavior of the quadratic variation $V_N(1)$ and of its conditional expectation given the σ-algebra generated by W. It is well-known that (see e.g. Protter (2005))

$$\sqrt{N}\left(V_N(1) - \int_0^1 (B_s^H)^2 ds\right)$$

converges in distribution to $\sqrt{2}\int_0^1 (B_s^H)^2 dW_s'$ where W' is a Wiener process independent by B_s^H (and W). By the same argument used in Remark 1, it is clear that such a result cannot be obtained if we replace $\int_0^1 (B_s^H)^2 ds$ by its expectation which is $\frac{1}{2H+1}$. We will analyze the asymptotic normality of the conditional expectation of $V_N(1)$ given $\mathcal{F}^W$. By Proposition 2, it follows that for every N the random variable $I\!E\left[V_1 \mid \mathcal{F}_1^W\right]$ is a multiple integral in the second Wiener chaos. It is possible to characterize the convergence in distribution of a sequence of multiple integrals to the standard normal law. We will use the following result (see Theorem 4 in Nualart and Ortiz-Latorre (2008), see also Nualart and Peccati (2005)).

Theorem 2 *Fix $n \geq 2$ and let $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ (with $f_k \in L^2([0,1]^n)$ for every $k \geq 1$) be a sequence of square integrable random variables in the n-th Wiener chaos such that $I\!E[F_k^2] \to 1$ as $k \to \infty$. Then the following are equivalent:*

i) The sequence $(F_k)_{k\geq 0}$ converges in distribution to the normal law $\mathcal{N}(0,1)$.

ii) One has $I\!E[F_k^4] \to 3$ as $k \to \infty$.

iii) For all $1 \leq l \leq n-1$ it holds that $\lim_{k\to\infty} \|f_k \otimes_l f_k\|_{L^2([0,1])^{\otimes 2(n-l)}} = 0$.

iv) $\|DF_k\|^2_{L^2([0,1]^n} \to n$ in $L^2(\Omega)$ as $k \to \infty$, where D is the Malliavin derivative with respect to B.

Criterion (iv) is due to Nualart and Ortiz-Latorre (2008); we will refer to it as the Nualart–Ortiz-Latorre criterion (which in fact is a refinement of the main result in Nualart and Peccati (2005).

We introduce the sequence of random variables given by, for every $N \geq 1$,

$$F_N := \frac{I\!E\left[V_N(1) \mid \mathcal{F}_1^W\right] - \theta}{a_N} \tag{19}$$

where a_N, the L^2 norm was introduced in (17) and $\theta = \frac{1}{2H+1}$. The main result of this section is the following.

Theorem 3 *The random variable F_N defined in equation* (19) *converges in distribution, as $N \to \infty$, to a standard normal random variable.*

Proof: We apply the general Theorem 2. To this end we need to compute the Malliavin derivative of F_N and its norm. Using the multiple integral form (expression (16)), we can write

$$D_\alpha F_N = \frac{2}{a_N} \sum_{i=0}^{N-1} 1_{[\frac{i}{N}, \frac{i+1}{N}]}(\alpha) \int_{\frac{i}{N}}^{\frac{i+1}{N}} R_H(u, \alpha) dW_u$$

and thus,

$$\begin{aligned}
\|DF_N\|^2_{L^2([0,1])} &= \frac{4}{a_N^2} \int_0^1 \left(\sum_{i=0}^{N-1} 1_{[\frac{i}{N}, \frac{i+1}{N}]}(\alpha) \int_{\frac{i}{N}}^{\frac{i+1}{N}} R_H(u, \alpha) dW_u \right)^2 d\alpha \\
&= \frac{4}{a_N^2} \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} d\alpha \left(\int_{\frac{i}{N}}^{\frac{i+1}{N}} R_H(u, \alpha) dW_u \right) \\
&= \frac{4}{a_N^2} \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} d\alpha \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{\frac{i+1}{N}} R_H(u, \alpha) R_H(v, \alpha) dW_u dW_v \\
&\quad + \frac{4}{a_N^2} \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} d\alpha \int_{\frac{i}{N}}^{\frac{i+1}{N}} R_H^2(u, \alpha) du
\end{aligned}$$

where for the last equality we used the product formula for multiple stochastic integrals. Consequently,

$$\|DF_N\|^2_{L^2([0,1])} = \frac{4}{a_N^2} \sum_{i=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} d\alpha \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{\frac{i+1}{N}} R_H(u, \alpha) R_H(v, \alpha) dW_u dW_v + 2. \tag{20}$$

Let us compute the $L^2(\Omega)$ norm of the random variable

$$\|DF_N\|^2_{L^2([0,1])} - 2.$$

From (20) and the isometry property of multiple stochastic integrals (4),

$$\mathbb{E}\left(\|DF_N\|^2_{L^2([0,1])} - 2\right)^2 = \frac{32}{a_N^4}\sum_{i=0}^{N-1}\int_{[\frac{i}{N},\frac{i+1}{N}]^4} R_H(u,\alpha)R_H(v,\alpha)R_H(u,\beta)R_H(v,\beta)dudvd\alpha d\beta.$$

The above sum can be decomposed into 81 terms. It is not difficult to estimate each of this term, using calculations similar with the computation of a_N in Proposition 3. For example, the first summand is

$$\begin{aligned}
&\frac{2}{a_N^4}\sum_{i=0}^{N-1}\int_{[\frac{i}{N},\frac{i+1}{N}]^4} u^{2H}v^{2H}\alpha^{2H}\beta^{2H}dudvd\alpha d\beta \\
&= \frac{2}{a_N^4}\frac{1}{N^{4(2H+1)}}\sum_{i=0}^{N-1}\left((i+1)^{2H+1} - i^{2H+1}\right) = \mathcal{O}(N^{-4}).
\end{aligned}$$

In this way, all the summands appearing in the decomposition of $\mathbb{E}\left(\|DF_N\|^2_{L^2([0,1])} - 2\right)^2$ converge to zero as $N \to \infty$ and the conclusion follows from Theorem 2 point *iv)*. ■.

We cite the Delta theorem (DasGupta, 2008):

Theorem 4 (Delta theorem) *Let T_n be a sequence of statistics such that:*

$$\sqrt{n}(T_n - \theta) \xrightarrow{\mathcal{L}} N(0, \sigma^2(\theta)), \quad \sigma(\theta) > 0.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function differentiable at θ with $g'(\theta) \neq 0$. Then,

$$\sqrt{n}(g(T_n) - g(\theta)) \xrightarrow{\mathcal{L}} N(0, [g'(\theta)]^2\sigma^2(\theta))$$

Thus, applying the Delta theorem we obtain a consistent estimator for H and its rate of convergence.

Theorem 5 *As $N \to \infty$, we have the convergence in distribution*

$$\sqrt{N}\left(\frac{1}{2\mathbb{E}\left[V_N(1) \mid \mathcal{F}_1^W\right]} - \frac{1}{2} - H\right) \to N\left(0, \frac{(2H+1)^4}{4}\right).$$

Proof: This follows from Proposition 3, Theorem 3 and the above Delta Theorem. ■

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