

Solution:

① Let urn 1 contain x white balls and y black balls. Then urn 2 will have $m-x$ white balls and $n-y$ black balls.

Let $A = \{ \text{the final ball drawn is white} \}$

Then call $P^{x,y}(A)$ = the probability of A when there are x white and y black balls in the first urn

$$\begin{aligned} \text{then } P^{x,y}(A) &= \frac{1}{2} \frac{x}{x+y} + \frac{1}{2} \frac{m-x}{m-x+n-y} = \\ &= \frac{1}{2} \left(\frac{x}{x+y} + \frac{m-x}{m+n-x-y} \right) \end{aligned}$$

Notice that $P^{x,y}(A) = P^{m-x,n-y}(A)$. Thus, without loss of generality ~~without~~ we can assume that $x \leq m-x$. Or $x \leq \frac{m}{2}$

~~We wish to maximize~~ We wish to maximize $P^{x,y}(A)$ over x and y . We will first show that $P^{x,y}(A) \leq P^{x,0}(A)$, $(A) x \in \{1, 2, \dots, \lfloor \frac{m}{2} \rfloor\}$. Then we will show that $P^{x,0}(A) \leq P^{1,0}(A)$, ~~thus~~ thus, the best choice is to leave one urn with only 1 white ball.

To do so let:

$$g(y) = \frac{x}{x+y} + \frac{m-x}{m+n-x-y}; \text{ with } x \text{ fixed}; x \in \{1, 2, \dots, \lfloor \frac{m}{2} \rfloor\}$$

We want to find $\max_{y \in \{0, 1, \dots, n\}} g(y)$

$$\begin{aligned} \text{We have } \frac{\partial}{\partial y} g(y) &= -\frac{x}{(x+y)^2} + \frac{m-x}{(m+n-x-y)^2} = \\ &= \frac{-x(m+n-x-y)^2 + (m-x)(x+y)^2}{(x+y)^2(m+n-x-y)^2} \end{aligned}$$

So to study the sign of $g'(y)$ is the same as studying the sign of numerator above; that is:

$$\begin{aligned} -x(m+n)^2 + 2x(m+n)(x+y) - (x+y)^2 \cdot x + (m-x)(x+y)^2 = \\ = (m-2x)(x+y)^2 + 2x(m+n)(x+y) - x(m+n)^2. \end{aligned}$$

At this point we realize that this is a quadratic function in $x+y$ so make the change of variables: $x+y = z$. Notice that studying the sign of the above expression for $y \in [0, n]$ is the same as studying the sign of the expression:

$$(m-2x)z^2 + 2x(m+n)z - x(m+n)^2$$

when $z \in [x, x+n]$.

We know about this expression (are complex)

- If the roots do not exist then the expression is always positive thus $g(y)$ is increasing so it has its maximum at one of the endpoints (more specifically at $y=n$)

- If the roots exist (are real) then we have :

$$z_1 z_2 = -\frac{x(m+n)^2}{m-2x} < 0 \text{ thus one root must be negative and one positive.}$$

This fact allows us to conclude that we are in one of the following situations:

- $0 < z_2 \leq x \leq x+n$. In this case the expression in z is positive on $[x, x+n]$ so $g'(y)$ is positive on $[0, n]$ so the maximum is attained at one of the endpoints ($y=0$ or $y=n$)
- $0 < x \leq z_2 \leq x+n$. Here the expression is negative on $[x, z_2]$ and positive on $(z_2, x+n]$. So ~~is~~ the expression in y for some $y_2 \in (0, n)$ we have then: $g(y)$ decreasing on $[0, y_2]$ and increasing on $(y_2, n]$. So the maximum is again attained at one of the endpoints (this time could be $y=0$ or $y=n$)
- $0 < x < x+n \leq z_2$: By the same reasoning we will conclude that $g(y)$ is decreasing on $[0, n]$ so max is attained at the endpoints ($y=0$).

So in any case maximum is attained at the endpoints and we need to compare the values at $y=0$ and $y=n$ to determine which gives the bigger value.

$$P^{x,0}(A) = 1 + \frac{m-x}{m+n-x} \quad P^{x,n}(A) = \frac{x}{x+n} + 1$$

So when is $P^{x,0}(A) \geq P^{x,n}(A)$?

That is:

$$1 + \frac{m-x}{m+n-x} \geq 1 + \frac{x}{x+n} \quad (\Rightarrow) \cancel{mx+mn-x^2-nx} \geq mx+nx-x^2$$

or $m \geq 2nx$ or $m \geq 2x$, but we are in this hypothesis
so the answer is always!

We concluded that $P^{x,y}(A) \leq P^{x_0}(A)$.

Now look at $P^{x_0}(A) = \frac{1}{2} \left(1 + \frac{m-x}{m+n-x} \right)$

$$\text{Let } f(x) = \frac{m-x}{m+n-x}$$

$$f'(x) = \frac{-(m+n-x) - (-1)(m-x)}{(m+n-x)^2} = \frac{-m-n+x+m-x}{(m+n-x)^2} = -\frac{n}{(m+n-x)^2} \leq 0$$

$\Rightarrow f$ is decreasing so the maximum is attained for the smallest value of x ; that is:

$$P^{1,0}(A) \geq P^{x_0}(A)$$

qed.

(2) Let $X \sim \text{Uniform}[0,1]$. Then the 2 segments will be X and $1-X$. $L_1 = \max(X, 1-X)$

$$L_2 = \min(X, 1-X)$$

$$(a) P(L_1 \leq x) = P(\max(X, 1-X) \leq x) = P(X \leq x \text{ and } 1-X \leq x) = \\ = P(X \leq x \text{ and } X \geq 1-x)$$

~~If $x < 1-x$~~ if $x < 1-x$ we cannot have a number less than x and bigger than $1-x$ so the above probability is zero

$$\Rightarrow x < 1-x \Leftrightarrow x < \frac{1}{2} \Rightarrow P(L_1 \leq x) = 0$$

$$\text{If } x > \frac{1}{2} \text{ we have } P(L_1 \leq x) = \int_{-x}^x \mathbb{1}_{[0,1]}(y) dy$$

If $x \in [\frac{1}{2}, 1] \Rightarrow 1-x \in [0, \frac{1}{2}] \Rightarrow [1-x, x] \subset [0, 1] \Rightarrow$

$$P(L_1 \leq x) = \int_{1-x}^x 1 dy = x - 1 + x = 2x - 1$$

If $x > 1 \Rightarrow 1-x < 0 \Rightarrow [0, 1] \subset [1-x, x] \Rightarrow$

$$P(L_1 \leq x) = \int_0^1 1 dy = 1$$

$$\Rightarrow P(L_1 \leq x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ 2x-1 & \text{if } x \in [\frac{1}{2}, 1] \\ 1 & \text{if } x > 1 \end{cases}$$

b) $P(L_2 \leq x) = 1 - P(L_2 > x) = 1 - P(\max(X, 1-x) > x) =$
 $= 1 - P(X > x \text{ and } 1-x > x) = 1 - P(X > x \text{ and } X < 1-x)$

Using the same reasoning as above you will obtain

$$P(X > x \text{ and } X < 1-x) = \begin{cases} 0 & \text{if } x > \frac{1}{2} \\ 1-2x & \text{if } x \in [0, \frac{1}{2}] \\ 1 & \text{if } x < 0 \end{cases}$$

$$\text{So } P(L_2 \leq x) = \begin{cases} 1 & \text{if } x > \frac{1}{2} \\ 2x & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x < 0 \end{cases}$$

gad.

~~1-x < 0~~ $\Rightarrow 1-x < 0 \Rightarrow [0, 1] \subset [1-x, x]$

$$\Phi(1-x) - \Phi(x) = x - 1 + x = 2x - 1$$

~~if $x \in [0, \frac{1}{2}] \Rightarrow [1-x, x] \subset [0, 1]$~~

③ Note that if $T \leq 15$ min then $P(\text{meeting}) = 1$

Assume that $T > 15$ min. Let a be a time in minutes

Let X_1 = time of the first person (arrival time)

X_2 = arrival time of the second person.

then $X_1 \sim \text{Uniform}[a, a+T] \Rightarrow f_{X_1}(t) = \frac{1}{T}$ if $t \in [a, a+T]$

$X_2 \sim \text{Uniform}[a, a+T] \Rightarrow f_{X_2}(t) = \frac{1}{T}$ if $t \in [a, a+T]$

They will meet if $|T_1 - T_2| \leq 15$

If they arrive independently the joint distribution is:

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} \frac{1}{T^2} & \text{if } x_1 \in [a, a+T] \text{ and } x_2 \in [a, a+T] \\ 0 & \text{else} \end{cases}$$

$$\therefore P(\text{meet}) = P(|T_1 - T_2| \leq 15) = P(15 + T_1 \leq T_2 \leq 15 + T_1) =$$

$$= \int_a^{a+T} \int_{T_1-15}^{T_1+15} \frac{1}{T^2} \cdot \mathbb{1}_{[a, a+T]}(t_2) \cdot dt_2 dt_1$$

This could be solved probably with the help of a picture like
we did in class but ~~we have seen that way already~~ we have seen that way already

We could do it analytically too.

First analyze when this integral is zero.

if $t_1 - 15 > a + T$ or $t_1 > a + T + 15$ (impossible since $t_1 < a + T$)

or if $t_1 + 15 < a$ or $t_1 < a - 15$ (impossible since $t_1 > a$)

So this integral is never zero (which is kind of obvious from the problem)

In the integral we need to look to the intersection:

$$[t_1 - 15, t_1 + 15] \cap [a, a+T] \quad \text{when } t_1 \in [a, a+T]$$

but when $t_1 \in [a, a+T]$ $[t_1 - 15, t_1 + 15]$ spans $[a-15, a+T+15]$

So the first clue is: consider what happens when

$$t_1 \in [a, a+15] \Rightarrow t_1 - 15 \leq a$$

$$\text{also } t_1 + 15 \in [a+15, a+30]$$

thus now we need to see if $T < 30$? If it is we have determined the upper limit of integration. So we will have 2 cases:

$$\textcircled{1} \quad 15 \leq T < 30:$$

$$\begin{aligned} P(\text{meet.}) &= \int_a^{a+T} \int_{t_1-15}^{t_1+15} \frac{1}{T^2} \mathbf{1}_{\{a, a+T\}}(t_2) dt_2 dt_1 = \\ &= \int_a^{a+15} \int_{a-15}^{\min(t_1+15, a+T)} \frac{1}{T^2} dt_2 dt_1 + \int_{a+15}^{a+T} \int_{t_1-15}^{\min(t_1+15, a+T)} \frac{1}{T^2} dt_2 dt_1 = \end{aligned}$$

$\Leftrightarrow t_1 + 15 < a + T \Leftrightarrow t_1 < a + T - 15$
 how is $a + T - 15$ compared to $a + 15$? $a + T - 15 < a + 15 \Leftrightarrow T < 30$ true.)

$$\begin{aligned} &= \int_a^{a+T-15} \int_a^{t_1+15} \frac{1}{T^2} dt_2 dt_1 + \int_{a+T-15}^{a+15} \int_a^{a+T} \frac{1}{T^2} dt_2 dt_1 + \int_{a+15}^{a+T} \int_{t_1-15}^{a+T} \frac{1}{T^2} dt_2 dt_1 = \\ &= \int_a^{a+T-15} \frac{t_1+15-a}{T^2} dt_1 + \int_{a+T-15}^{a+15} \frac{1}{T} dt_1 + \int_{a+15}^{a+T} \frac{a+T-t_1+15}{T^2} dt_1 = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{T^2} \left(\frac{t_1^2}{2} + (15-a)t_1 \right) \Big|_a^{a+T-15} + \frac{1}{T} \left(a+15 - \cancel{a-T+15} \right) + \frac{1}{T^2} \left((a+T+15)t_1 - \cancel{-\frac{t_1^2}{2}} \right) \Big|_{a+15}^{a+T} \\
&= \frac{1}{T^2} \left(\frac{(a+T-15)^2}{2} + (15-a)(a+T-15) - \frac{a^2}{2} - (15-a)a \right) + \frac{30-T}{T} + \\
&\quad + \frac{1}{T^2} \left((a+T+15)(a+T) - \frac{(a+T)^2}{2} - (a+T+15)(a+15) + \frac{(a+15)^2}{2} \right) = \\
&= \frac{1}{T^2} \left(\frac{a^2 + 2a(T-15) + (T-15)^2}{2} + (15-a)(a+T-15-a) - \frac{a^2}{2} + (a+T+15)(a+T-a-15) \right. \\
&\quad \left. + \frac{a^2 + 30a + 15^2 - a^2 - 2aT - T^2}{2} \right) + \frac{30-T}{T} = \\
&= \frac{1}{T^2} \left(\frac{2aT - 30a + T^2 - 30T + 15^2 + 30a + 15^2 - 2aT - T^2}{2} + 15T - 15^2 - aT \right. \\
&\quad \left. + 15a + aT - 15a + T^2 - 15T + 15T - 15^2 \right) + \frac{30-T}{T} = \\
&= \frac{1}{T^2} \left(-15T + 15^2 + 15T - 2 \cdot 15^2 + T^2 \right) + \frac{30-T}{T} = \\
&= \frac{1}{T^2} (T^2 - 15^2) + \frac{30-T}{T} = \cancel{1} - \frac{15^2}{T^2} + \frac{30}{T} - \cancel{1} = \underline{\underline{\frac{15}{T} \left(2 - \frac{15}{T} \right) = \frac{30}{T} - \frac{15^2}{T^2}}}
\end{aligned}$$

Notice that it does not depend on a and at $T=15$ is one as it should.

② When $\underline{T > 30}$ we have that ~~$a+T+15 > a+15$~~ so the integrals become:

$$P(\text{rect}) = \int_a^{a+15} \int_a^{t_1+15} \frac{1}{T^2} dt_2 dt_1 + \int_{a+15}^{a+T-15} \int_{t_1-15}^{t_1+15} \frac{1}{T^2} dt_2 dt_1 +$$

$$+ \int_{a+T-15}^{a+T} \int_{t_1-15}^{a+T} \frac{1}{T^2} dt_2 dt_1 = -g$$

$$= \int_a^{a+15} \left(\frac{(t_1 + 15 - a)}{T^2} \right) dt_1 + \int_{a+15}^{a+T-15} \frac{30}{T^2} dt_1 + \int_{a+T-15}^{a+T} \frac{a+T-t_1+15}{T^2} dt_1 =$$

$$= \frac{1}{T^2} \left(\frac{(a+15)^2}{2} + (15-a)(a+15) - \frac{a^2}{2} - (15-a)a + (a+T+15)(a+T) - \frac{(a+T)^2}{2} - (a+T+15)(a+T-15) + \frac{(a+T-15)^2}{2} \right) + \frac{30}{T^2} (a+T-15 - a - 15) =$$

$$= \frac{1}{T^2} \left(\frac{a^2 + 30a + 15^2}{2} + 15^2 - a^2 - \frac{a^2}{2} - 15a + a^2 + (a+T)^2 + 15(a+T) - \frac{(a+T)^2}{2} - (a+T)^2 + 15^2 + \frac{(a+T)^2 - 30(a+T) + 15^2}{2} \right) + \frac{30}{T^2} (T+30) =$$

$$= \frac{1}{T^2} \left(15a + \frac{15^2}{2} + 15^2 - 15a + 15a + 15T + 15^2 - 15a - 15T + \frac{15^2}{2} \right) + \frac{30}{T^2} (T+30) = 3 \cdot \frac{15^2}{T^2} + \frac{30}{T} - \frac{30^2}{T^2} = \underline{\underline{\frac{30}{T} - \frac{15^2}{T^2}}}$$

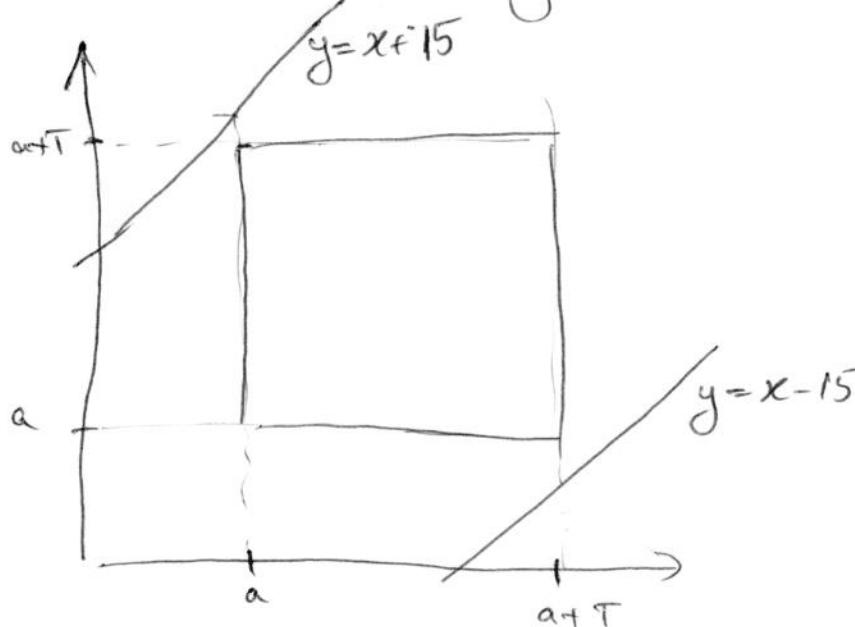
To verify note that $T=30$ gives the same value using both formulas ($= \frac{3}{4}$)

Thus: $P(\text{meet}) = \begin{cases} \frac{1}{2} & \text{if } T \leq 15 \\ \frac{30}{T} - \frac{15^2}{T^2} & \text{if } T > 15 \end{cases}$

③ geometrical solution:

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If $T < 15$ we have the following picture:



how do we know this? Notice that $y = x + 15$ intersects $x = a$ at $y = a + 15$
 since $T < 15 \Rightarrow a + T < a + 15 \Rightarrow$ left upper corner of ~~square is~~
 below the line $y = x + 15$

$\Rightarrow P(\text{meet}) = 1$ in this case.

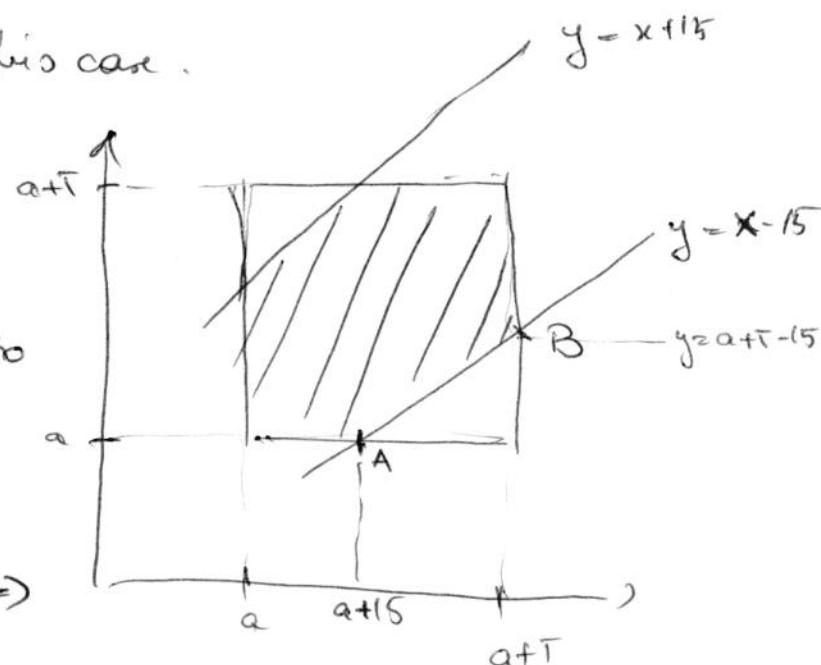
If $T > 15$ we have :

and we need the shaded area. To find it we need to calculate the points where they intersect.

$$\text{A has coordinates: } \begin{cases} y = x + 15 \\ y = a \end{cases} \Rightarrow$$

$$\rightarrow A = (a + 15, a)$$

$$\text{B has coordinates: } \begin{cases} y = x - 15 \\ x = a + T \end{cases} \Rightarrow B = (a + T, a + T - 15)$$



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Thus area of shaded region is:

$$\begin{aligned} & (a+T-a)(a+T-a) - 2 \cdot \left(\frac{a+T-15-a}{2} \right) \left(a+T-(a+15) \right) = \\ & = T^2 - 2 \cdot \frac{(T-15)(T-15)}{2} = T^2 - (T-15)^2 = 30T - 15^2 \end{aligned}$$

Thus $P(\text{meet}) = \frac{30T - 15^2}{T^2}$ if $T > 15$ and notice that we obtain the same answer as before

④ Take the transformation:

$$① (x, y) \rightarrow (x+y, y) = (u, v) \Rightarrow \begin{cases} y = v \\ x = u - v \end{cases}$$

$$\Rightarrow J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\Rightarrow f_{u,v}(u, v) = f(u-v, v)$$

$$\Rightarrow f_u(u) = \underbrace{\int_{-\infty}^{\infty} f(u-v, v) dv}_{\text{ }} \quad \text{ } \quad \text{ }$$

$$② (x, y) \rightarrow (x-y, y) = (u, v) \Rightarrow \begin{cases} y = v \\ x = u + v \end{cases} \Rightarrow J_{u,v} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\Rightarrow f_{u,v}(u, v) = f(u+v, v)$$

$$\Rightarrow f_u(u) = \underbrace{\int_{-\infty}^{\infty} f(u+v, v) dv}_{\text{ }}$$

$$③ (x, y) \rightarrow \cancel{(x, y)} (x/y, y) \Rightarrow \begin{cases} y = v \\ x = \frac{u}{v} \end{cases} \Rightarrow J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$\Rightarrow f_{u,v}(u, v) = \left| \frac{1}{v} \right| \cdot f\left(\frac{u}{v}, v\right)$$

$$\Rightarrow f_u(u) = \underbrace{\int_{-\infty}^{\infty} \left| \frac{1}{v} \right| f\left(\frac{u}{v}, v\right) dv}_{\text{ }}$$

$$\textcircled{4) d)} (x, y) \rightarrow \left(\frac{x}{y}, y \right) = (u, v) \rightarrow \begin{cases} y = v \\ x = uv \end{cases} \Rightarrow J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

$$\Rightarrow f_{u,v}(u, v) = |J| \cdot f(uv, v)$$

$$\Rightarrow f_u(u) = \int_{-\infty}^{\infty} |v| f(uv, v) dv$$

\textcircled{5)} Here anything will do. You have to give an example of 2 variables X and Y that have the property that $E(X)E(Y) = E(XY)$ and still X, Y are not independent.

Simplest idea is to notice that the equality will hold when $E(X) = 0$ and $E(XY) = 0$. ~~such that~~

Also if $X^2 = Y$ we need only to find a variable X such that $E(X) = 0$ and $E(X^3) = 0$. Clearly X and $Y = X^2$ are not independent.

$$\text{Take } X = \begin{cases} 1 & \text{prob } \frac{1}{2} \\ -1 & \text{prob } \frac{1}{2} \end{cases} \Rightarrow Y = X^2 = 1 \text{ with prob 1}$$

$$X \cdot Y = X^3 = X = \begin{cases} 1 & \text{prob } \frac{1}{2} \\ -1 & \text{prob } \frac{1}{2} \end{cases}$$

Thus $E(X) = 0$, $E(Y) = 1$, $E(XY) = 0$ so we have found X, Y not independent with property $E(XY) = E(X)E(Y)$