Homework 1 (mandatory)
Math 611 Probability

September 23, 2005

(1) An urn (let us call it Urn 1) contains w_1 white balls and b_1 black balls. Another urn contains w_2 white balls and b_2 black balls. A ball is drawn at random from each urn, then one of the two balls so obtained is chosen at random.

   (a) What is the probability that the ball drawn is white?

   (b) Given that the ball was white what is the probability it came from urn 1

(2) Prove the following:

   (a) *Total Probability Formula:* Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and A_1, A_2, $\ldots A_n$ a partition of Ω show that:

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B|A_i)\mathbf{P}(A_i) \quad , \forall\, B \in \mathcal{F}$$

   (b) *Bayes Formula:* With the same notation as in part (a):

$$\mathbf{P}(A_j|B) = \frac{\mathbf{P}(B|A_j)(A_j)}{\sum_{i=1}^{n} \mathbf{P}(B|A_i)(A_i)} \quad , \forall\, B \in \mathcal{F}, and\, j \in \{1, 2, \ldots, n\}$$

(3) Prove that if A and B are two independent sets in some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ then so are A^C and B^C (independent).

(*For Ph.D. students only -worth extra credit for Masters'*) Generalize to n sets using induction.

(4) At the end of a well known course the final grade is decided with the help of an oral examination. There are a total of m possible subjects listed on some pieces of paper. Of them n are generally considered “easy”. Each student enrolled in the class, one after another, draws a subject at random then presents it. Of the first two students who has the better chance of drawing a “favorable” subject?

(5) *Buffon's needle problem.* Suppose that a needle is tossed at random onto a plane ruled with parallel lines a distance L apart, where by a "needle" we mean a line segment of length $l \leq L$. What is the probability of the needle intersecting one of the parallel lines?

(6) Give an example of two distinct random variables with the same distribution function

(7) A random variable X has distribution function

$$F(x) = a + b \arctan \frac{x}{2} \quad , -\infty < x < \infty$$

Find:

(a) The constants a and b

(b) The probability density function of X

(8) What is the probability that two randomly chosen numbers between 0 and 1 will have a sum no greater than 1 and a product no greater than $\frac{15}{64}$?

(9) Let X be a random variable taking each of the values -2, -1,1, 2 with probability $\frac{1}{4}$, and let $Y = X^2$. Prove that X and Y (although obviously dependent) have correlation coefficient 0

(10) Prove that if X is a random variable such that $\mathbf{E}e^{aX}$ exists, where a is a positive constant, then:

$$\mathbf{P}\{X \geq \varepsilon\} \leq \frac{\mathbf{E}e^{aX}}{e^{a\varepsilon}}$$

Hint: Chebyshev's inequality may help here.

(11) Suppose the probability of hitting a target with a single shot is 0.003. What is the approximate probability $\mathbf{P}$ of hitting the target 15 or more times in 5000 shots?

(12) Suppose an event A has probability 0.3. How many trials must be performed to assert with probability 0.9 that the relative frequency of A differs from 0.3 by no more than 0.1.