# Homework 2
## *Math 611 Probability*

November 25, 2005

**NB:** This is an extra points assignment (it is not mandatory). There are 15 problems, the more you solve, the more points you may get. Its due date is TBA.

(1) We play a game which consists in tossing a coin with probability $p$ of landing a Head. If we obtain a head we draw a number at random in the interval $[1, 2]$ and we gain that number. If we get a Tail we lose \$1. Find the Distribution of $X$, the amount gained or lost.

(2) (Proof of the Monotone Convergence Theorem) If $f_n$ is a sequence of measurable positive functions such that $f_n \uparrow f$ then:

$$\int_\Omega f_n(\omega)\, \mathbf{P}(d\omega) \uparrow \int_\Omega f(\omega)\, \mathbf{P}(d\omega).$$

(3) If $X$ and $Y$ are independent random variables defined on $(\Omega, \mathcal{F}, \mathbf{P})$ with $X, Y \in L^1(\Omega)$, then $XY \in L^1(\Omega)$ and

$$\int_\Omega XY\, d\mathbf{P} = \int_\Omega X\, d\mathbf{P} \int_\Omega Y\, d\mathbf{P}$$

(**Hint** Use the standard approach *or* the Transport formula with $f = (X, Y)$ and $\psi(x, y) = |xy|$.

(4) If $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space, and $\mathcal{G}$ is a $\sigma$-sub-algebra of $\mathcal{F}$ prove using the definition of conditional expectation that:

(a) $\mathbf{E}[\mathbf{E}[X|\mathcal{G}]] = \mathbf{E}[X]$

(b) if $\mathcal{G}_1 \subset \mathcal{G}_2$, then

$$\mathbf{E}[\mathbf{E}[X|\mathcal{G}_1]|\mathcal{G}_2] = \mathbf{E}[\mathbf{E}[X|\mathcal{G}_2]|\mathcal{G}_1] = \mathbf{E}[X|\mathcal{G}_1]$$

(c) if $X$ is independent of $\mathcal{G}$ then: $\mathbf{E}[X|\mathcal{G}] = E[X]$

(d) If $Y$ is measurable with respect to $\mathcal{G}$ then $\mathbf{E}[XY|\mathcal{G}] = Y\mathbf{E}[X|\mathcal{G}]$

(5) A hen lays $N$ eggs where $N$ has the Poisson distribution with parameter $\lambda$. Each egg then hatches with probability $p$ independent of all the other eggs. Let $K$ be the number of chicks. Find the distribution of $K$, $\mathbf{E}(K|N)$, $\mathbf{E}(N|K)$, $\mathbf{E}[K]$

(6) Show that the generating function of a Poisson random variable with parameter $\lambda$ is $e^{\lambda(s-1)}$.

(7) If X has generating function $G(s)$ show that:

(a) $\mathbf{E}(X) = G^{(1)}(1)$

(b) $\mathbf{E}\left[X(X-1)\dots(X-k+1)\right] = G^{(k)}(1)$,
where $G^{(k)}(x)$ denotes the $k$-th derivative of $G$ at $x$.

(8) Show that if $X$ and $Y$ are independent then $G_{X+Y}(s) = G_X(s)G_Y(s)$, where $G$'s are generating functions.

(9) Show that if $X$ and $Y$ are independent then $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$, where $\phi$'s are characteristic functions.

(10) Show that if $X$ is a random variable and $Y = aX + b$ then we have the following relationship between the characteristic functions of the two variables:

$$\phi_Y(t) = e^{itb}\phi_X(at)$$

(11) (Stirling's formula) $n!$ $n^n e^{-n}\sqrt{2\pi n}$ is equivalent with:

$$\frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \to 1,$$

when $n \to \infty$. Prove this limit.

**Hint** The book on page 190 has this problem as an example. Make sure that you understand it.

(12) Show that if $X_n \to X$ in distribution and $g$ is a continuous function, then $g(X_n) \to g(X)$ in distribution.

(13) Give an example of a sequence of random variables $X_n$ that converges a.s. to a variable $X$ but $\mathbf{E}(X_n)$ does not converge to $\mathbf{E}(X)$. (a.s. convergence does not imply convergence in $L^1$).

(14) Show that $X_n \to 0$ in probability is equivalent with $\mathbf{E}\left(\frac{|X_n|}{1+|X_n|}\right) \to 0$

(15) Show that if $c$ is a constant and $X_n \to c$ in distribution then $X_n \to c$ in probability.