Ma 623 Stochastic Processes
Final Examination

DUE May 14, 2006 by 4:00pm

Please work independently on the following problems. No collaboration is permitted. You may use the textbook or any other research source you may find appropriate. You have to prove everything that you use, or quote the results you are applying.

(1) Let $X_t, t \geq 0$, be a Poisson process of rate λ. Let $T_0 = 0$, and let T_i be the time of the i^{th} observation (jump of X), if $i \geq 1$. Let $N = \inf\{k \geq 1 : T_k - T_{k-1} > 1\}$. Find $\mathbf{E}T_N$, $\mathbf{E}N$, and $\mathbf{E}(T_N|N = 8)$.

(2) Let X_t, $t \geq 0$, be a continuous time birth and death process which starts at 0, with parameters $\lambda > 0$ and $\mu > 0$, so that from any integer n the process can jump only to $n+1$ or $n-1$ and does so with rates λ and μ respectively. Let Z_n, $n \geq 1$, be random variables independent of X_t, $t \geq 0$, such that Z_n is exponential with mean n. Find sequences of numbers a_n and b_n (which may depend on λ and μ) such that

$$\frac{X_{Z_n} - a_n}{b_n}$$

converges in distribution to a distribution that does not put all its probability on a single point. Identify the limiting distributions as best you can.

(3) It's the year 2020. Professor Roger Pinkham is now living in Paris. Each weekday, he arrives for lunch at his favorite café at a random time which is uniformly distributed between 12 noon and 12:15 p.m. The time after his arrival until the waiter asks for his order is an independent random variable which is exponential with mean 5 minutes.

a) Given that Professor Pinkham is asked for his order at precisely 12:15 p.m., what is the conditional probability that he arrived before 12:05 p.m.?

b) Given that Professor Pinkham is asked for his order before 12:15 p.m., what is the conditional probability that he arrived before 12:05 p.m.?

(4) Suppose $\{X_n\}_{n\geq 0}$ is an irreducible, aperiodic Markov chain on a finite state space Σ. Define a Markov chain $\{Y_n\}_{n\geq 0}$ valued in $\Sigma \times \{0,1\}$ as follows:

$$\begin{cases} Y_n & = (X_n, 1) \text{ if } n \text{ is odd,} \\ Y_n & = (X_n, 0) \text{ if } n \text{ is even.} \end{cases}$$

a) Prove that $\{Y_n\}_{n\geq 0}$ is irreducible.

b) Given an example to show that $\{Y_n\}_{n\geq 0}$ need not be irreducible if $\{X_n\}_{n\geq 0}$ is periodic.

(5) Organisms are born at times $\{T_n\}_{n\geq 1}$, where $T_n = Y_1 + Y_2 + \ldots + Y_n$ and $Y_1, Y_2, \ldots$ are iid nonnegative r.v.s. with probability density function $g(x)$. The n^{th} organism (i.e., the organism born at time T_n) has a lifetime X_n. Assume that $X_1, X_2, \ldots$ are iid, strictly positive r.v.s. with probability density function $f(x)$, and that $X_1, Y_1, X_2, Y_2, \ldots$ are independent. Assume that

$$\begin{cases} \mu_X & = \int_0^\infty x f(x) dx < \infty \text{ and} \\ \mu_Y & = \int_0^\infty y g(y) dy < \infty. \end{cases}$$

Let $M(t)$ be the *expected* number of organisms alive at time t. Prove that

$$\lim_{t\to\infty} M(t)$$

exists, and find its value.

(6) Let $\{X_n\}$ be a Markov chain with state space $\{0, 1, 2, \ldots, K\}$ and the transition matrix

$$\begin{cases} \mathbf{P}(i, i+1) = \frac{2}{3} \\ \mathbf{P}(i, i-1) = \frac{1}{3} \end{cases}, \quad 1 \le i \le K-1; \quad \mathbf{P}(0,0) = \mathbf{P}(K,K) = 1.$$

Let $X_0 = i_0$ for some $1 \le i_0 \le K-1$. Let $\{X_n^*, n \ge 0\}$ be another process which is such that for any sequence of states $i_0, i_1, \ldots, i_n$,

$$\begin{cases} \mathbf{P}(X_0^* = i_0, X_1^* = i_1, \ldots, X_n^* = i_n) \\ = \mathbf{P}(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n | X_m = K \text{ eventually}). \end{cases}$$

a) Prove that $\{X_n^*\}$ is a Markov chain.

b) Find the transition matrix for $\{X_n^*\}$.

(7) Let $\{X_n\}_{n\ge 0}$, be a Markov chain on the state space $A = \{\ldots, \frac{1}{k^2}, \frac{1}{k}, 1, k, k^2, \ldots\}$, where $1 < k < \infty$. The transition probabilities are as follows:

$$\begin{cases} P(X_{n+1} = k^{m+1} | X_n = k^m) = p, \\ P(X_{n+1} = k^{m-1} | X_n = k^m) = 1 - p, \end{cases}$$

where $0 < p < 1$. Assume that $X_0 \equiv 1$.

a) For what values of p and k is the process X_n a martingale? supermartingale? submartingale?

b) What is the limiting behavior of X_n as $n \to \infty$? (In particular, does $X_n \to^{\text{a.s.}}$?)

(8) Let $\{B_t\}_{t\ge 0}$ be a standard Wiener process (Brownian motion). Define stopping times $\tau_a, a > 0$, by

$$\tau_a = \inf\{t \ge 1 : \; B_t^2 \ge at\}.$$

a) Prove that if $a > 1$ then $E\tau_a = \infty$.

b) Prove that if $a < 1$ then $E\tau_a < \infty$.

c) What is $E\tau_1$?

Hint: You may use the fact that $B_t^2 - t$ is a martingale.