Homework 1

Ma623 Stochastic Processes

due Tuesday Jan 31 2006

Bernoulli(p) process. Let $Y_1, Y_2, \ldots, Y_n$ be iid, Bernoulli(p) random variables, thus each variable takes the values 1 and 0 with probabilities p and $1-p$ respectivelly, with p a number between 0 and 1. Think of the Y_i's as outcomes from tossing a coin with probability p of showing a "Head". Define:

$$N_k = \text{Number of Heads in the first } k \text{ tosses.}$$

$$S_n = \inf\{k \mid N_k = n\} = \text{Time at which } n\text{-th H occurs}$$

$$X_n = S_n - S_{n-1} = \text{Number of tosses to get from } (n-1)\text{-st H to } n\text{-th H}$$

Remark: $N_k = \sup\{n \mid S_n \leq k\}$

(1) Find the joint probability:

$$\mathbf{P}\{X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4\},$$

for some x_1, x_2, x_3, x_4 positive integers.

(2) Find $\mathbf{P}\{S_5 = 10\}$

(3) Given that $N_{100} = 5$ find:

$$\mathbf{P}\{S_1 = 15, S_2 = 20, S_3 = 50, S_4 = 75, S_5 = 87\}$$

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(4) Given $S_4 = 100$, describe the joint distribution of $\{S_1, S_2, S_3\}$

(5) Show that $\mathbf{P}\{S_n > k\} = \mathbf{P}\{N_k < n\}$ for any n, k in $\mathbb{N}^*$.

(6) Prove that as $k \to \infty$; $\frac{N_k - kp}{\sqrt{kp(1-p)}} \xrightarrow{d} N(0,1)$

(7) Prove that as $k \to \infty$; $\frac{S_n - n/p}{\sqrt{n(1-p)/p}} \xrightarrow{d} N(0,1)$

(8) Prove that as $p \to 0$:

$$\mathbf{P}\{N_{[\frac{1}{p}t]} = j\} \to \frac{t^j}{j!}e^{-t}$$

Hint Usual proof that Binomial($n, p_n = \frac{\mu}{n}$) $\xrightarrow{d}$ Poisson (μ).