Ma 612 Mathematical Statistics
Midterm Examination

due in class April 16, 2006

(1) The random variables $X_1, X_2, \ldots, X_n$ are i.i.d., they take values in a finite set

$$A_m = \{1, 2, \ldots, m\}$$

and are uniformly distributed over A_m. The random variables $X_i's$ could be numbers that you see, e.g., the number of questions in an exam.

(a) Assume that m is not known and you want to estimate it. Write down the estimator obtained (i) by the method of moments and (ii) by maximizing the likelihood.

(b) If $\hat{m}$ is the MLE, show that $\hat{m}$ is biased and consistent for m, i.e., $E_m(\hat{m}) \neq m$, but $\hat{m} \to m$ if m is kept fixed and $n \to \infty$.

(c) Suppose m and n both tend to infinity at the same rate, i.e., $m = k \cdot n, n \to \infty$, with k a fixed constant. Show that $(\hat{m} - m)$ converges in distribution.

(2) Let the variables $X_1, X_2, \ldots, X_n$ i.i.d. from the distribution:

$$f_1(x|\theta_1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $\theta_1 = (\mu, \sigma^2)^T \in \mathbb{R} \times (0, \infty)$, and let $Y_1, Y_2, \ldots, Y_n$ i.i.d. from the distribution:

$$f_2(y|\theta_2) = \frac{1}{2\sqrt{2\pi\sigma_1^2}} e^{-\frac{(y-\mu_1)^2}{2\sigma_1^2}} + \frac{1}{2\sqrt{2\pi\sigma_2^2}} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}},$$

where $\theta_2 = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)^T \in \mathbb{R}^2 \times (0, \infty)^2$.

(a) Show that both functions above are legitimate pdf's

(b) Calculate the likelihood functions in both cases $L_1(\theta_1|x_1, \ldots, x_n)$, and $L_2(\theta_2|y_1, \ldots, y_n)$

(c) Show that the first likelihood function is always bounded on $\mathbb{R} \times (0, \infty)$, but the second is always unbounded on $\mathbb{R}^2 \times (0, \infty)^2$

(d) What does part (c) imply about the MLE estimators?

(3) Let $X_1, X_2, \ldots, X_n$ be Bernoulli random variables with the following joint distribution, depending on $\theta = (\theta_1, \theta_{11}, \theta_{10})$;

$$P(X_1 = 1) = \theta_1$$

$$P(X_i = 1|X_1, \ldots, X_{i-1}) = \begin{cases} \theta_{11} & \text{if} X_{i-1} = 1 \\ \theta_{10} & \text{if } X_{i-1} = 0 \end{cases}$$

for $i = 1, 2, \ldots, n$. Find a four dimensional sufficient statistic.

(4) Given a loss function $L(\theta, d)$ and a prior distribution π, the optimal prior decision is the d^π which minimizes $E^\pi[L(\theta, d)]$. That is, d^π is the decision minimizing the Bayes Risk with no observations.

The value of the sample information X=x is defined as

$$\nu(x) = E^\pi[L(\theta, d^\pi) \mid x] - E^\pi[L(\theta, \delta^\pi(x) \mid x]$$

where $\delta^\pi(x)$ is the Bayes rule corresponding to the prior π.

(a) Indicate why $\nu(x)$ is non-negative.

(b) If $X \mid \theta \sim N(\theta, \sigma^2)$, $\theta \sim N(\eta, \tau^2)$ and $L(\theta, d) = (\theta - d)^2$ what is the value of $\nu(x)$?

(5) Suppose X has density

$$f_\theta(x) = \begin{cases} (1 + \theta x)/2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

where the parameter space is $\{\theta : -1 \leq \theta \leq 1\}$ and θ has a prior distribution which is uniform on $[-1, 1]$.

(a) Find the Bayes estimate $\delta_1(x)$ of θ under squared error $L(\delta, \theta) = (\delta - \theta)^2$.

(b) Find the Bayes estimate $\delta_2(x)$ of θ under absolute error $L(\delta, \theta) = |\delta - \theta|$.

(c) Find the maximum likelihood estimate $\delta_3(x)$ of θ.

(d) For $x > 0$, the three estimates are ordered, $\delta_i(x) > \delta_j(x) > \delta_k(x)$. What is the order?

(6) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables uniform on $[0, \theta]$ with $\theta \geq 1$. We wish to estimate $g(\theta) = \theta^{-p}$ for $p > n$.

(a) Show that

$$T(X_{(n)}) = \begin{cases} 1 & X_{(n)} \leq 1 \\ X_{(n)} & X_{(n)} > 1 \end{cases}$$

is a sufficient statistic ($X_{(n)} = \max\{x_i\}$).

(b) Find the Best Unbiased Estimator (or UMVUE) of $g(\theta)$. Comment on the values of the estimates when $T \geq 1$.