CHAPTER 3

The Poisson process

I believe the treatment of the Poisson process is absolutely essential in modern stochastic processes treatment due to the vast array of applications of this process. We will start with basic definitions first.

3.1. Definitions.

DEFINITION 3.1 (Counting Process). N_t is a counting process if and only if

(1) $N_t \in \{0, 1, 2 \ldots\}, \forall t$
(2) N_t is non decreasing as a function of t

Here N_t is non-decreasing means that all the sample path $N_t(w)$ are non-decreasing as a function of t for every $w \in \Omega$, w fixed.

DEFINITION 3.2 (Poisson Process). $N(t)$ is a Poisson(λ) process if it is a counting process and in addition

(1) $N(0) = 0$.
(2) $N(t)$ has stationary independent increments.
(3) $P(N(h) = 1) = \lambda h + o(h)$.
(4) $P(N(h) \geq 2) = o(h)$.

Facts:

1: $f \sim o(g)$ if and only if $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$
2: $f \sim O(g)$ is and only if there exist c_1, c_2 constants, such that $c_1 \leq \frac{f(x)}{g(x)} \leq c_2$, $\forall x$ in a neighborhood of 0.

THEOREM 3.3. *If $N(t)$ is a Poisson(λ) process then*[1]

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

PROOF. A standard proof (presented in the notes handed in class) derive and solves the Kolmogorov's forward differential equations of the Poisson(λ) process, a discrete state space Markov Chain. This method

[1]Note that $P(N(t) = n) = P(N(s+t) - N(s))$ by the stationarity of the increments.

will be seen later and it is worth your time to read and understand that proof as well.

Here we will approach the proof a bit different. The idea is to approximate a Poisson(λ) process with a Bernoulli process and then pass to the limit.

Fix $t > 0$, cut $[0, t]$ into 2^k equally spaced intervals.

Let $\widetilde{N}_k$ = the number of these 2^k intervals with at least one event in them. Note that we have the condition

$$\widetilde{N}_k \leq N_t \quad \begin{cases} = & \text{only when each interval contain at most 1 event} \\ \leq & \text{always} \end{cases}$$

In addition, let

$$E_k = \{N_t > \widetilde{N}_k\} = \bigcup_{i=0}^{2^{k-1}} \left\{ \underbrace{N\left(\frac{i+1}{2^k}t\right) - N\left(\frac{i}{2^k}t\right) \geq 2}_{\text{at least 1 interval with 2 events}} \right\}$$

Take probability on both sides

$$\begin{aligned} P(E_k) &\leq \sum_{i=0}^{2^{k-1}} P\left(N\left(\frac{i+1}{2^k}t\right) - N\left(\frac{i}{2^k}t\right) \geq 2\right) \\ &= \sum_{i=0}^{2^{k-1}} P\left(N\left(\frac{1}{2^k}t\right) \geq 2\right) \qquad \text{(by stationarity)} \\ &= 2^k o\left(\frac{t}{2^k}\right) = \frac{o(t/2^k)}{t/2^k}\, t \xrightarrow{k\to\infty} 0, \end{aligned}$$

for every t fixed. So $\mathbf{P}(E_k) \xrightarrow{k\to\infty} 0$

Now $\widetilde{N}_k \sim$ Binomial$(2^k, \lambda\frac{t}{2^k} + 2o(\frac{t}{2^k}))$ Note that $\lambda\frac{t}{2^k} + 2o(\frac{t}{2^k})$ is the probability that at least one event occurs in an interval, i.e., $p = \mathbf{P}(N(\frac{t}{2^k}) = 1) + \mathbf{P}(N(\frac{t}{2^k}) \geq 2)$

Exercise 20. If $W_k \sim$ Binomial(k, p_k) and $kp_k \to \lambda$ when $k \to \infty$, then

$$W_k \xrightarrow{\mathcal{D}} \text{Poisson}(\lambda)$$

,i.e.

$$P(W_k = n) \to \frac{\lambda^n}{n!}e^{-\lambda}$$

In our case

$$2^k\left(\lambda\frac{t}{2^k} + 2o(\frac{t}{2^k})\right) = \lambda t + 2\frac{o(t/2^k)}{t/2^k}t \xrightarrow{k\to\infty} \lambda t$$

Therefore the Exercise 20 implies

$$\widetilde{N}_k \xrightarrow{\mathcal{D}} \text{Poisson}(\lambda t) \tag{3.1}$$

OR

$$P(\widetilde{N}_k = n) \to \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Furthermore, $N(t) = \widetilde{N}_k + \underbrace{N(t) - \widetilde{N}_k}$ and we know $P(\widetilde{N}_k \neq N(t)) = P(E_k) \to 0$ as $k \to \infty$. Therefore, we must have:

$$\widetilde{N}_k \xrightarrow{\mathcal{D}} N(t) \tag{3.2}$$

From (3.1) and (3.2) the limits of $\widetilde{N}_k$ must be the same thing. Done. □

3.2. Inter-arrival and waiting time

Let:

$$\begin{aligned}
X_1 &= \text{time of the first event} \\
X_2 &= \text{time between 1st and 2nd event} \\
&\vdots \\
X_n &= \text{time between (n-1)-th and n-th event}
\end{aligned}$$

Let S_i = time of the i-th event and notice that:

$$\begin{aligned}
S_i &= \sum_{j=1}^{i} X_j \\
S_n &= \inf\{t : N(t) \geq n\} = \inf\{t : N(t) = n\}
\end{aligned}$$

Proposition 3.4. *$X_1, X_2 \ldots$ are iid random variable, exponentially distributed with mean $\frac{1}{\lambda}$.*

We will not prove this proposition instead we will prove the following claim:

Claim–Evidence: The distribution of S_n is Gamma(n, λ) or the p.d.f. of S_n is given by:

$$f_{S_n}(t) = \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} \qquad t \geq 0$$

We note that the exponential distribution is a special case of Gamma distribution. In fact, Exponential(λ) = Gamma$(1, \lambda)$. This is a useful fact to know since the Gamma distribution has some nice property, one of them being that if the two variables added are independent then:

$$\text{Gamma}(\alpha_1, \beta) + \text{Gamma}(\alpha_2, \beta) \stackrel{\mathcal{D}}{=} \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

For this reason if the Proposition 3.4 is true then we must have the distribution of the arrival times S_n as:

$$\begin{aligned} S_1 &= X_1 \sim \text{Gamma}(1, \lambda) \\ S_2 &= X_1 + X_2 \sim \text{Gamma}(2, \lambda) \\ &\vdots \\ S_n &= X_1 + \ldots + X_n \sim \text{Gamma}(n, \lambda), \end{aligned}$$

Therefore, proving the claim adds evidence in favor of the Proposition 3.4. In fact we will prove the Proposition using the claim.

PROOF OF THE CLAIM-EVIDENCE. We know that: $\{S_n \leq t\} = \{N(t) \geq n\}$ (convince yourself of the truth of this affirmation). Thus the c.d.f

$$F_{S_n}(t) = \mathbf{P}\{S_n \leq t\} = \mathbf{P}\{N(t) \geq n\} = \sum_{j=n}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t}$$

Take the derivative with respect to t: $\frac{\partial}{\partial t}$

$$\begin{aligned} f_{S_n}(t) &= \sum_{j=n}^{\infty} \left[\left(\frac{\lambda j (\lambda t)^{j-1}}{j!} \right) e^{-\lambda t} + \frac{(\lambda t)^j}{j!} (-\lambda) e^{-\lambda t} \right] \\ &= \lambda e^{-\lambda t} \sum_{j=n}^{\infty} \left[\frac{(\lambda t)^{j-1}}{(j-1)!} - \frac{(\lambda t)^j}{j!} \right] \\ &= \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} \quad \text{and DONE} \end{aligned}$$

OR using another way:

$$\begin{aligned} f_{S_n}(t) dt &= \mathbf{P}(t \leq S_n \leq t + dt) \\ &= P\left(\underbrace{N(t) = n-1}_{\text{independent}} \text{ and } \underbrace{\text{at least one event in } [t, t+dt]}_{\text{independent}} \right) \\ &= P(N(t) = n-1) P(N(dt) \geq 1) \\ &= \frac{(\lambda t)^{nt} e^{-\lambda t}}{(n-1)!} [\lambda dt + o(dt) + o(dt)] \end{aligned}$$

Dividing the last expression by dt and taking $dt \to 0$,

$$\begin{aligned} f_{S_n} &= \frac{(\lambda t)^{nt} e^{-\lambda t}}{(n-1)!}\left(\lambda + 2\frac{o(dt)}{dt}\right) \\ &\xrightarrow{dt\to 0} \frac{(\lambda t)^{nt} e^{-\lambda t}}{(n-1)!} \end{aligned}$$

□

The plan is to finish the proof of Proposition 3.4 by calculating the joint density of X_i's. To do so we need the joint density of the S_i's. To this end we will introduce the concept of *order statistic*.

3.2.1. Order Statistic. Let $Y_1 \dots Y_n$ be n random variables. We say that $Y_{(1)} \dots Y_{(n)}$ are the order statistics corresponding to $Y_1 \dots Y_n$ if $Y_{(k)}$ is the k-th smallest value among $Y_1 \dots Y_n$

LEMMA 3.5. *If Y_i's are continuous random variables with p.d.f. f then the joint density of the order statistics $Y_{(1)} \dots Y_{(n)}$ is given by*

$$f(Y_1 \dots Y_n) = n! \prod_{i=1}^{n} f(Y_{(i)})$$

PROOF. exercise. See page 66 of the handed notes. □

THEOREM 3.6. *The joint density of $(S_1 \dots S_n) = (X_1, X_1+X_2 \dots \sum_{i=1}^n X_i)$ is*

$$f_{S_1 \dots S_n}(t_1 \dots t_n) = \lambda^n e^{-\lambda t_n} I_{\{0 \le t_1 < t_2 \dots < t_n\}}$$

PROOF. Let $0 \le t_1 < t_2 \dots < t_n$, and $\delta > 0$ small enough[2] such that $0 \le t_1 < t_1 + \delta < t_2 < t_2 + \delta \dots < t_n$. Let

$$I_j = (t_j, t_j + \delta)$$

Goal: Find $\mathbf{P}(S_1 \in I_1, S_2 \in I_2 \dots S_n \in I_n)$, then take $\delta \to 0$ to obtain the joint density. Note that terms can be express as follows

$$\begin{cases} S_1 \in I_1 & \text{no event in } (0, t_1) \text{ and 1 event in } I_1 \\ S_2 \in I_2 & \text{no event in } (t_1 + \delta, t_2) \text{ and 1 event in } I_2 \\ \quad\vdots & \\ S_n \in I_n & \text{no event in } (t_{n-1} + \delta, t_n) \text{ and at least 1 event in } I_n \end{cases}$$

[2]In other words, chose δ such that we create non-overlapping intervals.

Then

$$\begin{aligned}
&\mathbf{P}(S_1 \in I_1, S_2 \in I_2 \dots S_n \in I_n) \\
&= \underbrace{e^{-\lambda t}}_{0\in(0,t_1)} \underbrace{\left(\frac{\lambda\delta}{1!} e^{-\lambda(t_2-t_1-\delta)}\right)}_{1\in I_1} \underbrace{e^{-\lambda t}}_{0\in(t_1+\delta,t_2)} \underbrace{\left(\frac{\lambda\delta}{1!} e^{-\lambda(t_2-t_1-\delta)}\right)}_{1\in I_2} \dots e^{-\lambda(t_n-t_{n-1}-\delta)} \quad \underbrace{(1-e^{-\lambda\delta})}_{\text{at least 1 in } I_n} \\
&= (\lambda\delta)^{n-1} e^{-\lambda\delta(n-1)} (1-e^{-\lambda\delta}) e^{-\lambda t_n} e^{\lambda(n-1)\delta} \\
&= (\lambda\delta)^{n-1} e^{-\lambda t_n} (1-e^{-\lambda\delta})
\end{aligned}$$

Divide[3] by δ^n

$$\begin{aligned}
\frac{\mathbf{P}(S_1 \in I_1, S_2 \in I_2 \dots S_n \in I_n)}{\delta} &= e^{-\lambda t_n} \lambda^{n-1} \underbrace{\frac{1-e^{-\lambda\delta}}{\delta}}_{\to\lambda} \\
&\to \lambda^n e^{-\lambda t_n}
\end{aligned}$$

$\square$

3.2.2. Finishing the proof.

Proof of Proposition 3.4. Note that $X_1 = S_1, X_2 = S_2 - S_1, \dots, X_n = S_n - S_{n-1}$. Therefore we can obtain their distribution from:

$$f_{X_1 \dots X_n}(x_1, \dots, x_n) = f_{S_1 \dots S_n}\left(x_1, x_1 + x_2, \dots, \sum_{i=1}^n x_i\right) |J| \mathbf{1}_{\{0 \le x_1 \le x_1 + x_2 \dots \le \sum_{i=1}^n x_i\}}$$

The determinant of the Jacobian J of the transformation is

$$\begin{vmatrix} 1 & 1 & 1\dots & 1 \\ 0 & 1 & 1\dots & 1 \\ & \vdots & \vdots & 1 \\ 0 & 0 & 0\dots & 1 \end{vmatrix} = 1$$

Hence

$$\begin{aligned}
f_{X_1 \dots X_n}(x_1 \dots x_n) &= f_{S_1,\dots,S_n}\left(x_1, x_1 + x_2, \dots, \sum_{i=1}^n x_i\right) \mathbf{1}_{\{0 \le x_1 \le x_1 + x_2 \dots \le \sum_{i=1}^n x_i\}} \\
&= \lambda^n e^{-\lambda(x_1 + x_2 \dots x_n)} \prod_{i=1}^n \mathbf{1}_{\{x_i \ge 0\}} \\
&= \prod_{i=1}^n \lambda e^{-\lambda x_i} \mathbf{1}_{\{x_i \ge 0\}}
\end{aligned}$$

which is the product of n independent exponential distributions. $\square$

[3] Note that $\frac{1-e^{-a}}{a} \xrightarrow{a \to 0} = 1$

COROLLARY 3.7. Given $S_n = t_n$ the other $n-1$ arrival times $S_1, S_2 \dots S_{n-1}$ have the same distribution as the order statics corresponding to $(n-1)$ independent uniform random variables on the interval $(0, t_n)$.

PROOF.

$$\begin{aligned} f_{S_1 \dots S_{n-1}|S_n}(t_1 \dots t_{n-1}|t_n) &= \frac{f_{S_1 \dots S_n}(t_1 \dots t_n)}{f_{S_n}(t_n)} \\ &= \frac{\lambda^n e^{-\lambda t_n} \mathbf{1}_{\{0 \le t_1 < t_2 \dots < t_n < t\}}}{\frac{\lambda e^{-\lambda t_n}(\lambda t_n)^{n-1}}{(n-1!)}} \\ &= \frac{(n-1)!}{t_n^{n-1}} \mathbf{1}_{\{0 \le t_1 < t_2 \dots < t_n < t\}} \end{aligned}$$

□

COROLLARY 3.8. Given $N(t) = n$ the n arrival times $S_1, S_2 \dots S_n$ have the same distribution as the order statics corresponding to n independent uniform random variables on the interval $(0, t)$, i.e.

$$f_{S_1 \dots S_n|N(t)}(t_1 \dots t_n|n) = \frac{n!}{t^n} \mathbf{1}_{\{0 \le t_1 < t_2 \dots < t_n < t\}}$$

PROOF. Exercise. □

PROPOSITION 3.9. *Assume that each event of a Poisson(λ) process can be classified as either Type I or Type II event. Furthermore, suppose that if an event occurs at time s then it is classified as being Type I with probability $p(s)$ and Type II with probability $1 - p(s)$.*

If $N_i(t)$ is the number of events of Type i, $i \in \{I, II\}$ by time t, then $N_1(t)$ and $N_2(t)$ are independent Poisson random variables with means(rates) $\lambda t p$ and $\lambda t(1-p)$ respectively, where

$$p = \frac{1}{t} \int_0^t p(s) ds$$

PROOF. Omitted. □

COROLLARY 3.10. In general if $N(t)$ is poisson(λ) process and events can be categorized into some category type A independently of the original process, then if $N_A(t)$ is the number of events of type A by time t, then $N_A(t)$ is Poisson with rate $\lambda \cdot \int_0^t p_A(s) ds$, where $p_A(s)$ is the probability that one event occurring at time s is of type A.

Note that the original Poisson(λ) process has the mean $\mathbb{E}[N(t)] = \lambda t$.

For the process counting the events of type A, the rate (mean) can be written[4] as $\mathbb{E}[N_A(t)] = \underbrace{\lambda t}\cdot\underbrace{\frac{1}{t}\int_0^t p_A(s)ds}$.

3.3. General Poisson Processes

Definition 3.11. Let $\mathscr{X}$ be a set, $\mathscr{G}$ be a σ-field on $\mathscr{X}$. A counting process on $(\mathscr{X},\mathscr{G})$ is a stochastic process $\{N(A)\}_{A\in\mathscr{G}}$ with the following properties:

(i)$N(A) \in \{0,1,2\ldots\}$

(ii)$N(\bigcup_{i=1}^{\infty} A_i,\omega) = \sum_{i=1}^{\infty} N(A_i,\omega), \forall w \in \Omega$ and $A_1, A_2\ldots$ disjoint sets in $\mathscr{G}$

Definition 3.12 (General Poisson Process). Let $(\mathscr{X},\mathscr{G},\mu)$ be a measure space, A Poisson process on $(\mathscr{X},\mathscr{G})$ with intensity μ is a counting process $\{N(A)\}_{A\in\mathscr{G}}$ with

1. $N(A)$ is a poisson random variable with mean $\mu(A)$
2. Independent increments, i.e., if $A_1, A_2\ldots A_n$ are disjoint $\mathscr{G}$ sets in $\mathscr{X}$, then $N(A_1), N(A_2),\ldots N(A_n)$ are independent random variables.

Theorem 3.13. *Let $\{N(A)\}_{A\in\mathscr{G}}$ be a counting process on $(\mathscr{X},\mathscr{G})$. Let $\mu(A) = \mathbb{E}[N(A)]$ for $A \in \mathscr{G}$. If:*

1. *$N(\cdot)$ has independent increments (as before),*
2. *$\forall\epsilon > 0$, there exists $\delta > 0$ such that $\forall$ A with $\mathbb{E}[N(A)] < \delta$,*
$$\frac{\mathbf{P}(N(A)\geq 2)}{\mu(A)} < \epsilon$$
3. *If $x \in \mathscr{X}$ with $\mu(\{x\}) > 0$ then $N(\{x\}) \sim Poisson(\mu(\{x\}))$*

Then $\{N(A)\}_{A\in\mathscr{G}}$ is a Poisson variable with mean $\mu(A)$

Consequence: If $\{N(A)\}_{A\in\mathscr{G}}$ satisfies the above then

$$P(N(A) = k) = \frac{\mu(A)^k}{k!}e^{-\mu(A)}$$

Example 3.14 (Non-homogenous Poisson Process). This is a simple generalization of the regular Poisson process. The rate is a function of time $\lambda(t)$ instead of λt. In terms of the previous definition $\mathscr{X} = [0,\infty)$, $\mathscr{G} = \mathscr{B}([0,\infty))$ and $\mu(A) = \int_A \lambda(t)dt$. Notice that this process does not have stationary increments anymore.

[4]Note that the expectation is the product between the rate of the original Poisson(λ) process and the probability that the event is of type A.

EXAMPLE 3.15 (Compound Poisson Process). For each hit i for a Poisson(μ_1) process N on $(\mathscr{X}, \mathscr{G})$ attach the random variables Y_i iid with c.d.f. F, which give rise to the probability measure μ_2[5]. Then the process:

$$Z(A) = \sum_{i=1}^{N(A)} Y_i$$

is a process called the compound poisson process on $[0, \infty) \times \mathbb{R}$ with intensity $\mu = \mu_1 \times \mu_2$

This is the most general definition of the compound process. In the particular case when N is a regular Poisson process we obtain:

$$Z(t) = \sum_{i=1}^{N(t)} Y_i,$$

called the (simple) compound Poisson process.

PROPOSITION 3.16. *If λ is the rate for the Poisson process $N(t)$ and the variables Y_i have mean μ and variance ν^2 then:*

$$\mathbb{E}[Z(t)] = \lambda\mu t, \quad V[Z(t)] = \lambda(\nu^2 + \mu^2)t$$

As an example of occurrence of such a process imagine claims arriving at a health insurance agency, with the time of events modeled by the Poisson process, and with the amount of the claim given by the variables Y_i.

EXAMPLE 3.17. Consider a system with possible states $\{1, 2 \ldots\}$. Individuals enters the "system" according to a Poisson(λ) process. At any time after the entry, any individual is in some state $i \in \mathbb{N}^*$ ($\mathbb{N}^* = \mathbb{N} \setminus \{0\}$).

Let $\alpha_i(s) = \mathbf{P}\{\text{An individual is in state } i \text{ at time } s \text{ after entry}\}$.

Let $N_i(t)$ be the number of individual in state i at time t. Find $\mathbb{E}[N_i(t)]$.

SOLUTION: We can represent the state of each point of this process as the pair:

$$\left(\underbrace{\text{entry time}}_{\text{poisson process}}, \underbrace{\text{state at time t}}_{\text{the r.v., } Y_i} \right) \in [0, t] \times \mathbb{N}^*.$$

[5] One can obtain the measure from c.d.f. remembering that the Borel sets are generated by intervals and using the relation $\mu((a, b]) = F(b) - F(a)$

An event is of type i if at time t it is in state i. The process N that counts the number of particles on this set $\mathscr{X} = [0,t] \times \mathbb{N}^*$ is a general Poisson process.

Using this definition $N_i(t) = N([0,t] \times \{i\})$. Recalling the theorem 3.13, we have that $N_i(t)$ is a Poisson random variable with mean $\mu([0,t] \times \{i\})$. Therefore, the mean is:

$$\mu([0,t]\times\{i\}) = \lambda \int_0^t p(\text{event at time } s \text{ is of type } i) ds = \lambda \int_0^t \alpha_i(t-s) ds.$$

Let us look at this further. We have, using $r = t - s$:

$$\begin{aligned}
\lambda \int_0^t \alpha_i(t-s) &= \lambda \int_0^t \alpha_i(r) dr \\
&= \lambda \int_0^t \mathbf{P}(\text{invidual is in state } i,\, r \text{ units after its entry}) dr \\
&= \lambda \int_0^t \mathbb{E}\left[\mathbf{1}_{\{\text{individual is in state } i,\, r \text{ units after its entry}\}}\right] dr \\
\text{Fubini} \rightarrow &= \lambda \mathbb{E}\left[\underbrace{\int_0^t \mathbf{1}_{\{\dots\}} dr}_{\text{time spent in state } i \text{ during } [0,t]}\right] \\
&= \lambda \mathbb{E}\left[\text{time spent in state } i \text{ during the first } t \text{ time units}\right]
\end{aligned}$$

Question: What happens as $t \to \infty$? □

Example 3.18 (text 2.22). Cars enter a highway (one way highway) according to a poisson(λ) process in time. Each car has velocity $v(i)$ iid with c.d.f.$=F$.
Q: Assuming that each car travels at constant velocity, find the distribution of the number of cars on the highway between points a and b (spatial points) at time t?

Solution: We have the entry time and velocity, i.e.,

$$(\text{entry time, velocity}) = (S(i), v(i)) \in [0,\infty) \times [0,\infty)$$

A sample outcome is presented in Figure 1. The position of the car i at time t is $v(i)(t - S(i))$

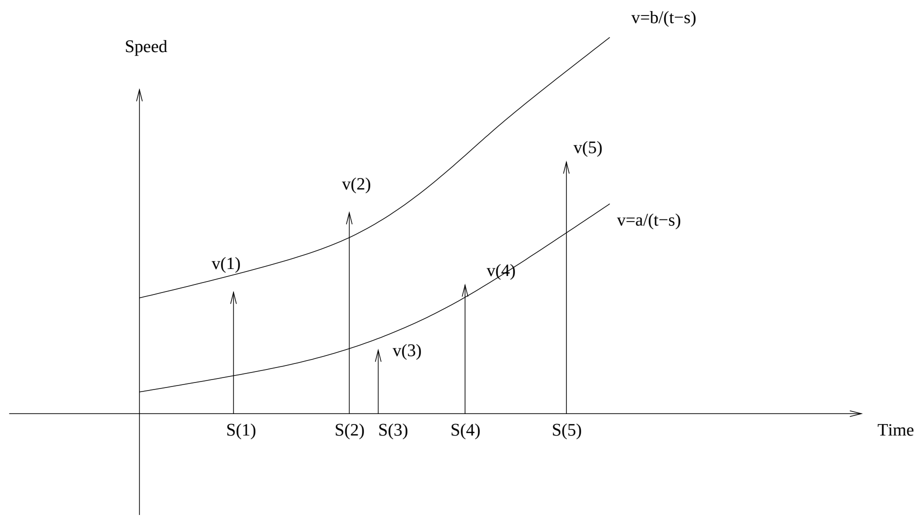

FIGURE 1. Cars enter at $S(i)$ with velocity $v(i)$

We call the event i an event of Type AB if a car entering at s_i with velocity v_i is in $[a, b]$ at time t. Then using Corollary 3.10:

$N(t)$ = number of cars in $[a, b]$ = number of events of type AB

$N(t) \sim$ Poisson with the rate $= \lambda \int_0^t p(\text{events enter at } s \text{ is of type } AB) ds$

What is the probability that a car that arrives at s will be in the interval $[a, b]$ at time t?

$$\begin{aligned} P(\{v : a < v \cdot (t-s) < b\}) &= P\left\{v : \frac{a}{t-s} < v < \frac{b}{t-s}\right\} \\ &= \left[F\left(\frac{b}{t-s}\right) - F\left(\frac{a}{t-s}\right)\right] \mathbf{1}_{[0,t]}(s) \end{aligned}$$

Therefore, $N(t)$ is a Poisson random variable with mean

$$\lambda \int_0^t \left[F\left(\frac{b}{t-s}\right) - F\left(\frac{a}{t-s}\right)\right] ds.$$

Question: What happens as $t \to \infty$? □

3.4. Simulation Techniques. Constructing the Poisson Process.

There are two ways to construct a 1-dim Poisson process

Simplest way : Let $X_1, X_2 \ldots$ iid, exponential(λ) with mean $\frac{1}{\lambda}$. Use X_i as the time between events $i - 1$ and i. (Done!)

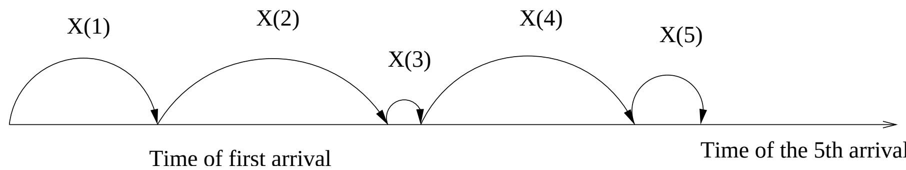

FIGURE 2. Illustration of the construction idea

Generate interval wise: For each time interval $[0,1), [1,2)\dots[t-1,t)\dots$

(1) Simulate[6] $N_I = N([k-1,k))$ = number of events in $I = [k-1,k)$, this generates the number of events in each interval.
(2) To get the actual times of the events, use the uniform distribution to generate times in each interval. For example if say you obtained $N([0,1)) = 2$ just generate 2 Uniform$(0,1)$ random variables, they are your 2 event times.

As anything in life, the simple way is simple but only works with the 1-dim process. The interval-wise way on the other hand is more complicated but it can be extended to the general Poisson process. The way to do it is straight forward. Suppose we have $(\mathscr{X},\mathscr{G})$ a measurable space, and μ a σ-finite measure (see Definition 1.11). Partition $\mathscr{X}$ into $\{B_i\}_{i=1}^{\infty}$ such that $\mu(B_i) < \infty$. Then for each B_i get:

(1) $N(B_i)$ = the number of events in B_i which is distributed as a Poisson$(\mu(B_i))$ random variable,
(2) $X_1^{(i)}, X_2^{(i)}\dots$ iid[7] with probability distribution

$$P(X_k^{(i)} \in A) = \mu(A|B_i) = \frac{\mu(A\cap B_i)}{\mu(B_i)}$$

Then for every $A \in \mathscr{G}$ let

$$N(A) = \sum_{i=1}^{\infty} N(A\cap B_i) = \sum_{i=1}^{\infty}\left[\sum_{k=1}^{\infty}\mathbf{1}_{\{X_k^{(i)}\in A \text{ and } N(B_i)\geq k\}}\right] \tag{3.3}$$

THEOREM 3.19. *The construction above and* (3.3) *yields a Poisson process with intensity* μ *on* $(\mathscr{X},\mathscr{G})$.

SKETCH OF THE PROOF. We omit the detailed proof, but we give bellow the important ideas of the proof.

[6] Note that for each interval, $N([k-1,k))$ are iid Poisson$(\lambda\cdot 1)$ random variables.
[7] random points positions in B_i

The countable additivity is satisfied automatically, by the definition of measure. The proof continues demonstrating the following facts:

(1) $N(A) \sim \text{Poisson}(\mu(A))$ for any $A \in \mathscr{G}$
(2) $N(A)$ and $N(B)$ are independent if $A \cap B = \emptyset$

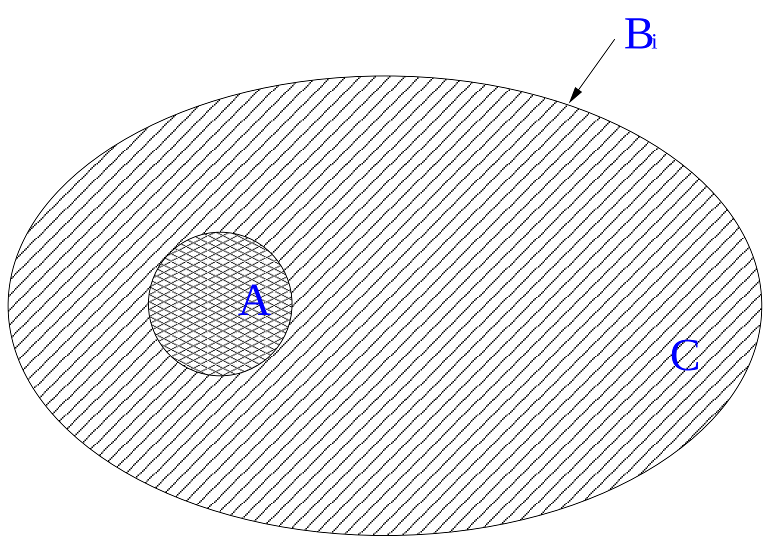

FIGURE 3. Illustration of set A and B_i

First show these properties inside each B_i. To show (2) inside B_i we may proceed as follows. Let $A \subset B_i$ and $C = B_i \backslash A$ (see Figure 3). For integers $a, c \in \mathbb{N}$ we have:

$$
\begin{aligned}
\mathbf{P}(N(A) = a, & N(C) = c) = \mathbf{P}(N(A) = a, N(C) = c, N(B_i) = a + c) \\
&= \mathbf{P}(N(A) = a, N(C) = c | N(B_i) = a + c)\mathbf{P}(N(B_i) = a + c) \\
&= P(N(A) = a | N(B_i) = a + c) \cdot P(N(B_i) = a + c) \qquad (3.4) \\
&= P(N(A) = a | N(B_i) = a + c) \cdot \frac{[\mu(B_i)]^{a+c}}{(a+c)!} e^{-\mu(B_i)} \qquad (3.5) \\
&= \binom{a+c}{a} [\mu(A|B_i)]^a [\mu(C|B_i)]^c \cdot \frac{[\mu(B_i)]^{a+c}}{(a+c)!} e^{-\mu(B_i)} \qquad (3.6) \\
&= \binom{a+c}{a} \frac{[\mu(A \cap B_i)]^a [\mu(c \cap B_i)]^c}{(a+c)!} e^{-\mu(B_i)} \qquad (3.7) \\
&= \frac{[\mu(A \cap B_i)]^a [\mu(c \cap B_i)]^c}{a!c!} e^{-\mu(A \cap B_i) - \mu(C \cap B_i)} \\
&= \underbrace{\frac{[\mu(A \cap B_i)]^a}{a!} e^{-\mu(A \cap B_i)}}_{\text{poisson in } A} \cdot \underbrace{\frac{[\mu(C \cap B_i)]^c}{c!} e^{-\mu(C \cap B_i)}}_{\text{poisson in } C} \\
&= \frac{[\mu(A)]^a}{a!} e^{-\mu(A)} \cdot \frac{[\mu(C)]^c}{c!} e^{-\mu(C)} \\
&= P(N(A) = a) \cdot P(N(C) = c)
\end{aligned}
$$

In (3.4), we removed the redundant information.
In (3.5), we used the Poisson distribution to write the probability for $P(N(B_i) = a + c)$
In (3.6) we used the binomial distribution with $n = a + c$ and $p = \mu(A|B_i)$
In (3.7) by the definition of conditional probability $[\mu(A|B_i)]^a = \frac{[\mu(A \cap B_i]^a}{[\mu(B_i]^a}$

Therefore, $N(A)$ and $N(C)$ are independent. □

Example 3.20 (Astronomy). Consider stars distributed in space according to a 3D Poisson process with intensity, $\lambda\mu$, where μ is the Lebesgue measure[8] on $\mathbb{R}^3$, $\lambda > 0$. Let x, y be 3-dim vectors (position).

[8]The Lebesgue measure is the standard way of assigning a length, area or volume to subset of Euclidean space. It is used throughout real analysis, in particular to define Lebesgue integration. Sets which can be assigned a volume are called Lebesgue measurable; the volume or measure of the Lebesgue measurable set A is denoted by $\lambda(A)$. A Lebesgue measure of ∞ is possible, but even so, assuming the axiom of choice, not all subset in $\mathbb{R}^n$ are Lebesgue measurable. The "strange" behavior of non-measurable sets gives rise to such statements as the Banach-Tarski paradox, a consequence of the axiom of choice.

Assume that light intensity exerted at x by a star located at y is

$$f(x,y,\alpha)=\frac{\alpha}{\|x-y\|^2}=\frac{\alpha}{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2},$$

where α is a random parameter depending on the size of the star at y.

Assume that α's associated with stars are iid with common mean μ_α and common variance σ_α^2. Also assume that the combined intensity at x accumulates additively.

Let $Z(x,A)$ be the total intensity at x due to stars in the region A. Then:

$$Z(x,A)=\sum_{i=1}^{N(A)}\frac{\alpha_i}{\|X-Y_i\|^2}=\sum_{i=1}^{N(A)}f(x,y_i,\alpha_i),$$

where $N(A)$ is the number of stars in the region A in space. Note that Y and α are random variables.

We have:

$$\mathbb{E}[Z(x,A)]=\mathbb{E}[N(A)]\mathbb{E}[f(x,Y,\alpha)]. \tag{3.8}$$

We do not prove this result here but note that the expression is a direct consequence of the Wald's equation.

We have that $\mathbb{E}[N(A)]=\lambda\mu(A)$, where $\mu(A)$ is the volume of A.

$$\mathbb{E}[f(x,Y,\alpha)]=\mathbb{E}\left[\frac{\alpha}{||x-Y||^2}\right]=\mathbb{E}[\alpha]\mathbb{E}\left[\frac{1}{||x-Y||^2}\right]$$

Since α and Y are independent. As a consequence of the Poisson Process in space, Y is going to be uniform in A or: $\mathbb{E}[||x-Y||^{-2}]=\frac{1}{\mu(A)}\int_A\frac{1}{||x-y||^2}dy$, then applying the equation (3.8) we have:

$$\begin{aligned}\mathbb{E}[Z(x,A)]&=\lambda\mu(A)\mu_\alpha\frac{1}{\mu(A)}\int_A\frac{1}{||x-y||^2}dy\\&=\lambda\mu_\alpha\int_A\frac{1}{||x-y||^2}dy\end{aligned}$$

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