CHAPTER 4

# Renewal Processes

In the past I have concentrated on these processes because of the strength of the theorems one can prove. I will state here the majority of the results but we are going to cover a lot less than what this chapter contains.

EXAMPLE 4.1 (Typical example where renewal process appears). A light bulb in a room keeps burning out. Assume that a mechanism instantaneously replaces the bulb with another one as soon as it burns. Describe the Number of light bulbs replaced by time $t$.

Let $X_1, X_2 \dots$ iid with c.d.f. $F$, $X_i$ positive, (non identical zero), with $\mathbb{E}[X_1] = \mu \in (0, \infty]$. These variables will describe the lifetimes of the light bulbs. Define:

$$S_n = \text{time to replace the } n\text{-th bulb}$$
$$S_n = \sum_{i=1}^{n} X_i \sim \underbrace{F * F \dots F}_{n \text{ times}} = F_n$$

Note that $F_n$ means $F$ convoluted[1] itself $n$ times.
We define the renewal process, $N(t)$ as:

$$\begin{aligned} N(t) &= \sup\{n : S_n \le t\} \\ &= \text{number of renewals up to time } t \end{aligned}$$

Note that a Poisson($\lambda$) process is a renewal process. In that special case the $X_i$'s are exponentially distributed. For a general renewal process, $X_i$'s could have any distribution.
We have the property: $\{N(t) \ge n\} \Leftrightarrow \{S_n \le t\}$. This is the same result we had for the Poisson($\lambda$) process. Therefore, we can write:

$$\begin{aligned} P(N(t) = n) &= P(N(t) \ge n) - P(N(t) \ge n+1) \\ &= P(S_n \le t) - P(S_{n+1} \le t) = F_n(t) - F_{n+1}(t) \end{aligned}$$

[1]Recall $X, Y \sim F, G$ and with pdf $f, g$ then: $X + Y \sim F * G(z) = G * F(z) = \int_{-\infty}^{\infty} f(x)g(z-x)dx$

**The renewal function** The renewal function is the main topic of our study.

$$
\begin{aligned}
m(t) = \mathbb{E}[N(t)] &= \text{The expected number of renewals by time } t \\
m(t) = \mathbb{E}[N(t)] &= \mathbb{E}\left[\sum_{i=1}^{\infty} I_{\{S_i \leq t\}}\right] \\
&= \sum_{i=1}^{\infty} \mathbb{E}\left[I_{\{S_i \leq t\}}\right] \\
&= \sum_{i=1}^{\infty} P(S_i \leq t) \\
&= \sum_{i=1}^{\infty} F_i(t)
\end{aligned}
$$

Thus we just showed that:

$$
m(t) = \sum_{i=1}^{\infty} F_i(t) \tag{4.1}
$$

PROPOSITION 4.2. *$m(t) < \infty$, for all $0 < t < \infty$ fixed*

PROOF. Assume $P(X_k \geq 1) = p > 0$. We will make this assumption. Since $P(X_k = 0) < 1$ then it must exist a positive value $\alpha$ such that $P(X_k \geq \alpha) = p > 0$. If the proof works with $\alpha = 1$ we can later substitute $\alpha$ and the proof will not change significantly.

Let $j - 1 \leq t \leq j$

**Claim:** $N(t)$ the number of renewals by time $t \leq$ sum of $j$ independent "total" number of trials Geometric($p$) random variable. Let us prove the claim. For each bulb $k$,

$$
\begin{aligned}
&\text{If } X_k < 1 = \text{throw away the bulb (it counts as a renewal)} \\
&\text{If } X_k \geq 1 = \text{use the bulb for 1 unit of time then throw it away}
\end{aligned}
$$

If $N^*(t)$ is the number of bulbs replaced by time $t$ using the protocol described above, we obviously have $N^*(t) \geq N(t)$. This proves the claim since $N^*(t)$ has the desired probability distribution.

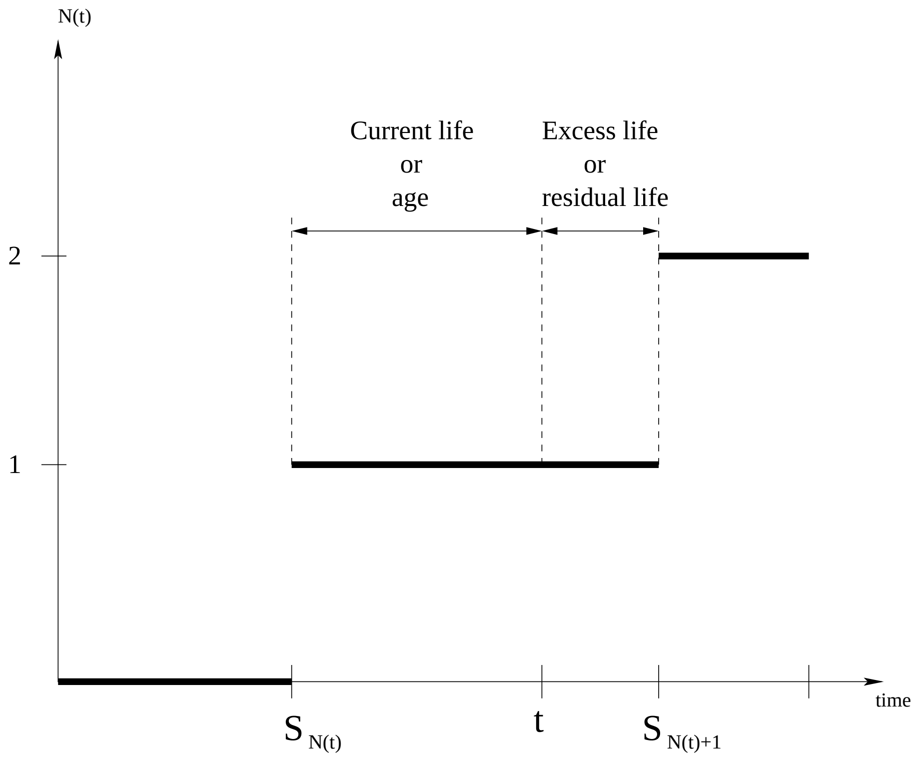


FIGURE 1. Relationship between $S_{N(t)}$, $t$, and $S_{N(t)+1}$

Therefore, using the claim:

$$
\begin{aligned}
m(t) = \mathbb{E}[N(t)] &\leq \mathbb{E}[N^*(t)] \\
&= \mathbb{E}[Y_1 + Y_2 \dots Y_j] \\
&= \underbrace{\frac{1}{p} + \frac{1}{p} \dots \frac{1}{p}}_{j \text{ times}} \\
&= \frac{j}{p} < \frac{t+1}{p} < \infty \quad (\text{because } j - 1 < t < j)
\end{aligned}
$$

We also have:

$$
\begin{aligned}
\mathbb{E}[N(t)^2] &\le \mathbb{E}[N^*(t)^2] \\
&\le \mathbb{E}[(Y_1 + Y_2 \dots Y_j)^2] \\
&= \underbrace{\mathrm{Var}(Y_1 + Y_2 \dots Y_j)}_{\text{negative binomial}} + \underbrace{(\mathbb{E}[Y_1 + Y_2 \dots Y_j])^2}_{\text{known}} \\
&= \frac{j(j-p)}{p^2} + \left(\frac{t+1}{p}\right)^2 < c(t+1)^2 < \infty \quad \text{as well.} \qquad (4.2)
\end{aligned}
$$

□

## 4.1. Limit Theorems for the renewal process

We will consider limiting results (as $t \to \infty$) for the processes defined thus far.

Proposition 4.3 (Strong Law of Large Numbers for renewal processes). *Using the notation defined earlier,*

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \textit{a.s. as } t \to \infty$$

Proof of SLLN:. $\frac{N(t)}{t} \to \frac{1}{\mu}$ a.s.
Recall that $S_n = \sum_{i=1}^{n} X_i$. Then the regular SLLN for random variables gives that $S_n \Rightarrow \frac{S_n}{n} \to \mu$ a.s.

By the definition of $N(t)$ we have $S_{N(t)} \le t < S_{N(t)+1}$

Divide both sides by $N(t)$

$$\underbrace{\frac{S_{N(t)}}{N(t)}}_{\to\mu \text{ by SLLN}} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} = \underbrace{\frac{S_{N(t)+1}}{N(t)+1}}_{\to\mu \text{ by SLLN}} \cdot \underbrace{\frac{N(t)+1}{N(t)}}_{\to 1 \text{ a.s.}}$$

which implies $\frac{t}{N(t)} \to \mu$ a.s. OR $\frac{N(t)}{t} \to \frac{1}{\mu}$ a.s.

□

Now we want to obtain a convergence result for $m(t)$. Notice that $m(t) = \mathbb{E}[N(t)]$ and we already have a convergence result for $N(t)$ (this SLLN). Can we get a result about $m(t)$ immediately. Not necessarily as the following example shows.

Example 4.4 (a.s. convergence does not necessarily imply $L^1$-convergence). Assume that $X_n \xrightarrow{\text{a.s.}} 0$. Is is always true that $\mathbb{E}[X_n] \to 0$?

ANSWER: Not necessary, for example let $U \sim$ Uniform$[0,1]$, and define $X_n = n\mathbf{1}_{\{U<\frac{1}{n}\}}$. Then we have $X_n \xrightarrow{\text{a.s.}} 0$, but

$$\mathbb{E}[X_n] = n \cdot P\left(\mu < \frac{1}{n}\right) = n \cdot \frac{1}{n} = 1 \to 1 \neq 0$$

□

However for our particular case the implication is true. For the result to be true we need to apply either the dominated convergence theorem or the monotone convergence theorem.

THEOREM 4.5 (Elementary renewal theorem). *With the earlier notations we have:*

$$\frac{m(t)}{t} \to \frac{1}{\mu} \text{ as } t \to \infty$$

*with the convention* $\frac{1}{\infty} = 0$

UNIMAGINATIVE PROOF. Recall that we showed in (4.2) that $\mathbb{E}[N(t)^2] \leq c(t+1)^2$. Thus we have:

$$\mathbb{E}\left[\left(\frac{N(t)}{t}\right)^2\right] \leq \frac{c(t+1)^2}{t^2} \leq 2c \quad \text{which is independent of } t$$

Therefore, $\frac{N(t)}{t}$ is uniformly integrable (since it is in $L^2$). Thus we get the desired result immediately □

**4.1.1. Wald's Theorem. Discrete stopping time.** We could just leave the Elementary renewal theorem the way it is, after all we have proven it. However, instead we will prove it again using different concepts which we will use latter on.

The first such new concept is the next theorem which is very very general and very, very useful.

THEOREM 4.6 (Wald's Theorem/Identity/Equation). *Let $X_1, X_2 \ldots$, $W_1, W_2 \ldots$ be 2 sequence of random variables with $X_k$ independent of $W_k$ for any fixed $k$. If one of the following conditions is true*

(1) *All $X_k$'s and $W_k$'s are $\geq 0$*

(2) $\sum_{k=1}^{\infty} \mathbb{E}[W_k X_k] < \infty$

*Then*

$$\mathbb{E}\left[\sum_{k=1}^{\infty} W_k X_k\right] = \sum_{k=1}^{\infty} \mathbb{E}[W_k]\mathbb{E}[X_k]$$

Proof of Wald's Theorem: **If the first hypothesis is true** we have:

$$\mathbb{E}\left[\sum_{k=1}^{\infty} X_k W_k\right] \underbrace{=}_{\text{positivity}} \sum_{k=1}^{\infty} \mathbb{E}[X_k W_k]$$

$$\underbrace{=}_{\text{independence}} \sum_{k=1}^{\infty} \mathbb{E}[X_k]\mathbb{E}[W_k]$$

**If the second hypothesis is true:** $\sum_{k=1}^{\infty} \mathbb{E}[X_k W_k] < \infty$.

Let

$$\begin{aligned} W_k^+ &= W_k I_{\{W_k \geq 0\}} \\ W_k^- &= -W_k I_{\{W_k < 0\}} \\ X_k^+ &= X_k I_{\{X_k \geq 0\}} \\ X_k^- &= -X_k I_{\{X_k < 0\}} \end{aligned}$$

Note that

$$\begin{aligned} W_k &= W_k^+ - W_k^- \\ X_k &= X_k^+ - X_k^- \end{aligned}$$

We then have the following:

$$\sum_{k=1}^{\infty} W_k X_k = \sum_{k=1}^{\infty} W_k^+ X_k^+ - \sum_{k=1}^{\infty} W_k^+ X_k^- - \sum_{k=1}^{\infty} W_k^- X_k^+ + \sum_{k=1}^{\infty} W_k^- X_k^-$$

All $X_k^+$, $X_k^-$, $W_k^+$ and $W_k^-$ are positive, then from part 1, we have

$$\begin{aligned} \mathbb{E}[\sum W_k^+ X_k^+] &= \sum \mathbb{E}[W_k^+]\mathbb{E}[X_k^+] \\ \mathbb{E}[\sum W_k^+ X_k^-] &= \sum \mathbb{E}[W_k^+]\mathbb{E}[X_k^-] \\ \mathbb{E}[\sum W_k^- X_k^+] &= \sum \mathbb{E}[W_k^-]\mathbb{E}[X_k^+] \\ \mathbb{E}[\sum W_k^- X_k^-] &= \sum \mathbb{E}[W_k^-]\mathbb{E}[X_k^-] \end{aligned}$$

Recombining the terms in the expression above will finish the proof. □

Example 4.7. Let $X_1, X_2 \ldots$ iid with $\mathbb{E}[X_i] = \mu$. Define $X_k$ to be the gain at some game if you actually make the $k$-th bet. Let

$$W_k = \begin{cases} 1 & \text{if you win} \\ 0 & \text{if you loose} \end{cases}$$

then $\sum_{k=1}^{\infty} X_k W_k$ is the total gain from all bets.

Assume that $X_k > 0$, also assume that $W_k$ is determined by previous bets and by $X_1 \dots X_{k-1}$[2] and maybe some $U \sim$ uniform random variable Let $N = \sum_{i=1}^{\infty} W_k$ to be the number of bets you win. Then Wald's theorem says that if $\mathbb{E}[N] < \infty$ we have

$$\mathbb{E}\left[\sum_{k=1}^{\infty} W_k X_k\right] = \sum_{k=1}^{\infty} \mathbb{E}[W_k] \underbrace{\mathbb{E}[X_k]}_{=\mu}$$

$$= \mu \sum_{k=1}^{\infty} \mathbb{E}[W_k] = \mu \mathbb{E}[N]$$

**Note:** Think about this and explain to yourself why this result is obvious.

The second important concept is the notion of a stopping time defined next.

Definition 4.8 (Discrete Stopping time). Let $X_1, X_2 \dots$ a sequence of independent random variable. $N \in \{0, 1, 2 \dots\}$ is called a stopping time for this $\{X_n\}_n$ sequence if $\{N = n\}$ is independent of $X_{n+1}, X_{n+2} \dots$. Note that this is true if $\{N = n\}$ is determined only by $X_1, X_2, \dots, X_n$ (or $\{N = n\}$ is measurable with respect to the sigma algebra generated by $X_1, X_2, \dots, X_n$)

Corollary 4.9 (A simpler version of Wald's theorem which we will use for the renewal processes). . For $X_1, X_2 \dots$ iid with $\mu = \mathbb{E}[X_i]$ finite and $N$ a stopping time with $\mathbb{E}[N] < \infty$, then

$$\mathbb{E}[\sum_{k=1}^{N} X_k] = \mathbb{E}[X_i]\mathbb{E}[N] = \mu \mathbb{E}[N]$$

Proof. We wish to apply the general Wald. For this purpose notice that we can write: $\sum_{k=1}^{N} X_k = \sum_{k=1}^{\infty} X_k \mathbf{1}_{\{N \geq k\}}$. In order to apply regular Wald we need to show that $\mathbf{1}_{\{N \geq k\}}$ is independent of $X_k$.

Remark 4.10. $N$ is a stopping time $\Leftrightarrow$ $\{N \leq n\}$ is independent of $\{X_{n+1}, X_{n+2} \dots\}$.

Proof of this Remark: is an exercise. As a hint note that $\{N \leq n\} = \cup_{k=1}^{n} \{N = k\}$. $\square$

Then $\{N \leq n\}$ is independent of $X_{n+1}, X_{n+2} \dots$ by the remark
OR $\{N > n\}$ is independent of $X_{n+1}, X_{n+2} \dots$
OR $\{N > n - 1\}$ is independent of $X_n, X_{n+1} \dots$

[2] In other words, it can depend on previous wins but not on the current

which implies that $\{N \geq n\}$ is independent of $X_n, X_{n+1} \ldots \Rightarrow$ we can use Wald.

$$\begin{aligned}
\mathbb{E}\left[\sum_{k=1}^{N} X_k\right] &= \mathbb{E}\left[\sum_{k=1}^{\infty} X_k \mathbf{1}_{\{N \geq k\}}\right] \\
&= \sum_{k=1}^{\infty} \underbrace{\mathbb{E}[X_k]}_{=\mu} \mathbb{E}[\mathbf{1}_{\{N \geq k\}}] \\
&= \mu \sum_{k=1}^{\infty} \mathbb{E}[\mathbf{1}_{\{N \geq k\}}] \\
&= \mu \sum_{k=1}^{\infty} P(N \geq k) \\
&= \mu \begin{pmatrix} P(N=1) & +P(N=2) & +P(N=3) & +\ldots \\ & +P(N=2) & +P(N=3) & +\ldots \\ & & +P(N=3) & +\ldots) \end{pmatrix} \\
&= \mu[1 \cdot P(N=1) + 2 \cdot P(N=2) + 3 \cdot P(N=3) + \ldots] \\
&= \mu \mathbb{E}[N]
\end{aligned}$$

□

**4.1.2. Back to the renewal processes.** Now the idea is to use the Wald's theorem we just prove. To this end we need to find a stopping time for the inter-arrival times. So, the next question comes naturally: "Is $N(t) = \sup\{n : S_n \leq t\}$ a stopping time for $X_1 \ldots X_n \ldots$"

Short answer: No, since there could be an event happen between the $n$-th event and $t$.

Mathematical answer:

$$\begin{aligned}
\{N(t) = n\} &= \{S_n \leq t < S_{n+1}\} \\
&= \{X_1 + X_2 \ldots + X_n \leq t < \underbrace{X_1 + X_2 \ldots X_n + X_{n+1}}_{\text{not independent of } X_{n+1}}\}
\end{aligned}$$

Note that the event is determined by $X_{n+1}$, hence $\{N(t) = n\}$ can't be a stopping time.

New question: "Is $N(t) + 1$ a stopping time?"

Answer: Yes, note that

$$\begin{aligned}
\{N(t) + 1 = n\} = \{N(t) = n - 1\} &= \{S_{n-1} \leq t < S_n\} \\
&= \{X_1 \ldots + X_{n-1} \leq t < X_1 \ldots X_{n-1} + X_n\}
\end{aligned}$$

Since everything inside the last $\{\cdot\}$ does not contain terms of type $n+1$ or larger, $\{N(t) + 1\}$ is a stopping time.

At this point we are in the position to give the alternate proof to the elementary renewal theorem. We will restate the theorem first.

THEOREM 4.11 (Elementary Renewal Theorem). *Let* $m(t) = \mathbb{E}[N(t)]$.

$$\frac{m(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu} \quad (= 0 \text{ if } \mu = \infty)$$

ALTERNATE PROOF TO THE ELEMENTARY RENEWAL THEOREM. Since $N(t)+1$ is a stopping time, from Wald we have

$$\mathbb{E}[S_{N(t)+1}] = \mathbb{E}[N(t)+1]\mathbb{E}[X] = \mu(m(t)+1) \tag{4.3}$$

**Claim 1.** $\liminf\limits_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$

CLAIM 1 PROOF: By definition, $S_{N(t)+1} > t \Rightarrow \mathbb{E}[S_{N(t)+1}] > t$. Then using (4.3) we obtain:

$$\mu(m(t)+1) > t$$
$$\frac{m(t)}{t} > \frac{1}{\mu} - \frac{1}{t}$$

Take lim inf on both sides

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geq \liminf_{t\to\infty} \left(\frac{1}{\mu} - \frac{1}{t}\right) = \frac{1}{\mu}$$

□

**Claim 2.** $\limsup \frac{m(t)}{t} \leq \frac{1}{\mu}$

CLAIM 2 PROOF: Fix $M > 0$ constant. Let

$$\bar{X}_k = \begin{cases} X_k & \text{if } X_k \leq M \\ M & \text{if } X_k > M \end{cases}$$

Let $\mu_M = \mathbb{E}[\bar{X}_k]$, $\bar{N}(t)$ to be the number of renewals up to $t$ with lifetimes $\bar{X}_k$.

Note that $\bar{N}(t) \geq N(t)$ (due to shorter life times[3])

$$\bar{m}(t) = \mathbb{E}[\bar{N}(t)] \geq m(t)$$

Now look at the Figure 2 on page 62 which represents the behavior of the new process at $t$. Note that since we bounded the interarrival times by $M$ we have:

$$\bar{S}_{\bar{N}(t)+1} > t$$
$$\bar{S}_{\bar{N}(t)+1} \leq t + M$$

[3]Life span is limited by the upper bound $M$.

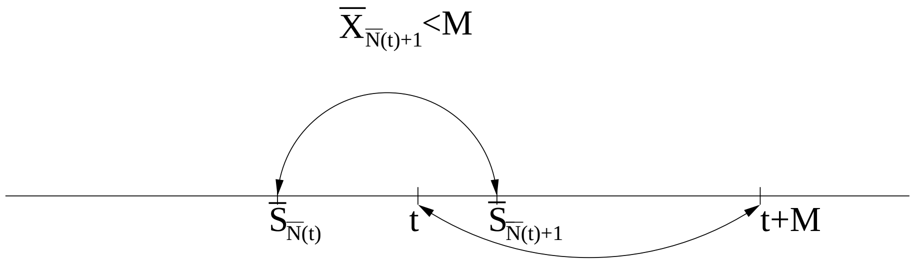


FIGURE 2. Representation of the new process.

If we use (4.3) for $\bar{N}(t)$ we get: $\mathbb{E}[\bar{S}_{\bar{N}(t)+1}] = \mu_M(\bar{m}(t)+1)$. Therefore:

$$\begin{aligned}\mu_M(\bar{m}(t)+1) &\leq t + M \\ \frac{\bar{m}(t)+1}{t} &\leq \frac{1}{\mu_M} + \frac{M}{t\mu_M} \\ \frac{\bar{m}(t)}{t} &\leq \frac{1}{\mu_M} + \frac{M}{t\mu_M} - \frac{1}{t}\end{aligned}$$

Apply lim sup in both sides to get:

$$\limsup_{t\to\infty} \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_M}$$

which implies

$$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \limsup_{t\to\infty} \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_M}.$$

But this holds for any $M > 0$. Therefore take $M \to \infty$ and using that $\lim_{M\to\infty} \frac{1}{\mu_M} = \frac{1}{\mu}$, we conclude: $\limsup\limits_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$. $\square$

Now combining the two claims will finish the proof. $\square$

Finally we will give a convergence in distribution result similar with the regular Central Limit Theorem.

THEOREM 4.12 (Renewal Central Limit Theorem). *Let $X_1, X_2 \ldots$ i.i.d., positive, $\mu = \mathbb{E}[X_k]$, $\sigma^2 = Var(X_k) < \infty$*

$$S_n = \sum_{i=1}^{n} X_i, \quad N(t) = \sup\{n : S_n \leq t\}$$

*Then*

$$P\left(\frac{N(t) - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} < y\right) \xrightarrow{t\to\infty} \Phi(y)$$

*where* $\Phi(y)$ *is the c.d.f. of* $N(0,1)$

In other words, when $t$ is large $N(t) \sim N(\frac{t}{\mu}, \frac{\sigma^2 t}{\mu^3})$

PROOF. Fix $y$.

$$P\left(\frac{N(t)-\frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} < y\right) = P\left(N(t) < \underbrace{\frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}}_{=r_t}\right)$$
$$= P(N(t) < r_t)$$
$$= P(S_{r_t} > t)$$

where it is obvious the definition of $r_t$. Note here that what we wrote only works if $r_t$ is an integer. If $r_t$ is not an integer, take $\tilde{r}_t = [r_t] + 1$. Then

$$P(N(t) < r_t) = P(N(t) < \tilde{r}_t) = P(S_{\tilde{r}_t} > t)$$

We can use the classic central limit theorem to complete the proof.

$$P(S_{\tilde{r}_t} > t) = P\left(\frac{S_{\tilde{r}_t} - \tilde{r}_t\mu}{\sigma\sqrt{\tilde{r}_t}} > \frac{t - \tilde{r}_t\mu}{\sigma\sqrt{\tilde{r}_t}}\right)$$
$$= \lim_{t\to\infty} \Phi\left(\frac{t-\tilde{r}_t\mu}{\sigma\sqrt{\tilde{r}_t}}\right)$$

The proof of the theorem will end if we show that $\frac{t-\tilde{r}_t\mu}{\sigma\sqrt{\tilde{r}_t}} \to -y$

We have that:

$$\tilde{r}_t = r_t + \underbrace{\{1 - \{r_t\}\}}_{=\Delta_t \in [0,1)}$$

where we used the notation $\{x\}$ for the fractional part of $x$. This implies:

$$\frac{t-\tilde{r}_t\mu}{\sigma\sqrt{\tilde{r}_t}} = \frac{t - r_t\mu - \Delta_t\mu}{\sigma\sqrt{r_t + \Delta_t}}$$

Recall that $r_t = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$. Therefore, we continue:

$$= \frac{t - \Delta_t\mu - \left(t + y\sigma\sqrt{\frac{t}{\mu}}\right)}{\sigma\sqrt{\Delta_t + \frac{t}{\mu} + \sigma y\sqrt{\frac{t}{\mu^3}}}} \xrightarrow{t\to\infty} \frac{-y\sigma}{\sigma} = -y$$

$\square$

## 4.2. Discrete Renewal Theory. Blackwell theorem.

We will start this section with a motivating example.

EXAMPLE 4.13 (Block Replacement Policy). Consider a light-bulb with lifetime $X$, a random variable. Due to economic reasons, it might be cheaper on a per bulb basis to replace all the bulbs instead of just the one that breaks. A block replace policy does just that by fixing a time period $K$ and replacing bulb's as they failed at times $1, 2 \ldots K-1$ and at $K$ replacing everything regardless the condition of the bulb. Let

$$c_1 = \text{replacement cost per bulb (block replacement)}$$
$$c_2 = \text{replacement cost per bulb (failure replacement)}$$

where obviously $c_1 < c_2$. Let $N(n)$ to be the number of replacements up to time $n$ for *1 bulb*, let $m(n) = \mathbb{E}[N(n)]$

For one bulb the expected cost is $c_2 m(k-1) + c_1$. Then, the mean cost per unit of time is:

$$\frac{\text{mean cost}}{\text{unit time}} = \frac{c_2 m(k-1) + c_1}{K}$$

Since the replacements take place only at the beginning of the day we are only interested in discrete variables to describe the lifetime of a lightbulb. Suppose that $X$ has the distribution $P(X = k) = p_k$, $k = 1, 2 \ldots$. Fix $n \leq K$. Look at $X_1$ the lifetime of the first lightbulb. Obviously, if $X_1 > n$ there was no replacement by time $n$. If $X_1 = k \leq n$ then we will have $m(n-k)$ expected replacements in the later time period. Therefore, we can write conditioning on the lifetime of the first bulb:

$$\begin{aligned} m(n) &= \sum_{k=n+1}^{\infty} p_k \cdot 0 + \sum_{k=1}^{n} p_k[1 + m(n-k)] \\ &= \sum_{k=1}^{n} p_k[1 + m(n-k)] \\ &= F_X(n) + \sum_{k=1}^{n-1} p_k m(n-k), \end{aligned}$$

where $F_X(\cdot)$ is the c.d.f. of $X$. Then we obtain recursively:

$$\begin{aligned} m(0) &= 0 \\ m(1) &= F_X(1) + p_1 m(0) = p_1 \\ m(2) &= F_X(2) + p_1 m(1) + p_2 m(0) = p_1 + p_2 + p_1^2 \\ &\vdots \quad \text{etc.} \end{aligned}$$

Example 4.14 (continues the example above). Let us look to a numerical example of the problem above. Suppose that $X$ can only take values $\{1, 2, 3, 4\}$ with $p_1 = 0.1$, $p_2 = 0.4$, $p_3 = 0.3$, $p_4 = 0.2$, furthermore the costs are $c_1 = 2$, $c_2 = 3$. Find the optimal replacement policy.

Using the formulas above we can calculate:

$$m(1) = 0.1, \quad m(2) = 0.51, \quad m(3) = 0.891, \quad m(4) = 1.3231$$

Using these numbers we will try to minimize the expect cost,

$$\text{cost} = \frac{c_1 + c_2 m(K-1)}{K} \leftarrow \text{We will try different K's to get the minimum}$$

We will obtain a table of cost as a function of K as:

Table 1. default

| $K$ | cost |
|---|---|
| 1 | 2.00 |
| 2 | 1.15 |
| 3 | 1.17 |
| 4 | 1.16 |
| 5 | 1.19 |

Hence the optimal replacement policy is at $K = 2$. We can also continue the calculation of $m$'s:

$$\begin{aligned} m(5) =& 1.6617, m(6) = 2.0647, m(7) = 2.4463, m(8) = 2.8336, \\ m(9) =& 3.2136, m(10) = 3.6016, \ldots \end{aligned}$$

Now we can calculate $u_n$ the probability that a replacement occurs in period $n$ as:

$$u_n = m(n) - m(n-1).$$

Calculating $u_n$'s for the values given we can see that pretty quickly we have

$$u_n \approx \frac{1}{\mu} = 0.3846.$$

This fact will be explained by the next theorem.

Let us assume that we have a renewal process with non negative integer valued lifetimes, $X$ with $P(X = k) = p_k$, $k = 0, 1, 2 \ldots$

Definition 4.15. $X$ an integer random variable is called a *lattice* if there $\exists d \geq 0$ such that $p_k > 0, \forall k$ not a multiple of $d$. The largest $d$ with the property that $\sum_{n=1}^{\infty} p_{nd} = 1$ is called the period of $X$. In effect:

$$d = \text{g.c.d.}\{k : p_k > 0\}$$[4]

If g.c.d.$\{k : p_k > 0\} = 1$ then $X$ is called a non lattice random variable. Also if $X$, a lattice random variable has c.d.f $F$, then $F$ is called a lattice.

Example 4.16. Consider the two simple example below

- $p_2 = p_4 = p_6 = \frac{1}{3}$ lattice distribution.
- $p_3 = p_7 = \frac{1}{2}$ non-lattice distribution.

In the previous example we have seen how to establish the equation

$$m(n) = F_X(n) + \sum_{k=1}^{n-1} p_k m(n-k).$$

(Note that if lifetimes are allowed to be zero the equation is a little different.)

However, this equation constitutes a particular example of a *renewal equation* (discrete case). In general a discrete renewal equation looks like:

$$v_n = b_n + \sum_{k=0}^{n} p_k v_{n-k}, \tag{4.4}$$

where $v_i$'s are unknowns and $p_i$'s are probabilities. Note that this form of equation has a unique solution, e.g. $v_0 = \frac{b_0}{1-p_0}$, $v_1 = \frac{b_1 + p_1 v_0}{1-p_0}$, etc.

Let $u_n$ be the expected number of renewals that take place in period $n$. We have said in the example that $u_n = m(n) - m(n-1)$. This is only true if lifetimes are nonzero and therefore at most one renewal occurs in any 1 time period. This is easy to show:

$$\begin{aligned} u_n &= \mathbf{P}\{\text{One renewal occurred at } n\} \\ &= \mathbb{E}[\mathbf{1}_{\{\text{One renewal occurred at } n\}}] \\ &= \mathbb{E}[N(n) - N(n-1)] = m(n) - m(n-1) \end{aligned}$$

We have seen in the previous example that this $u_n$ got closer and closer to $1/\mu$. The next theorem formalizes this fact and generalizes it.

[4]The greatest common denominator of the set of integers.

THEOREM 4.17 (Blackwell renewal theorem). *Using the notations defined thus far we have:*

(1) $u_n \to \frac{1}{\mu}$ *as* $n \to \infty$.
(2) *If* $X_0 \geq 0$ *is a "delay" variable, and* $X_1, X_2, \ldots \geq 0$ *are i.i.d. lifetimes independent of* $X_0$ *with* $\mathbb{E}X_1 = \mu$ *and non-lattice distribution then:*

$$m(t+a) - m(t) \to \frac{a}{\mu}, \quad \text{as } t \to \infty.$$

*Note that* $m(t+a) - m(t)$ *is the expected number of renewals in the interval* $[t, t+a]$.
(3) *If* $X_i$*'s are lattice random variables with period* $d$*, and* $X_0 = 0$ *then:*

$$\mathbb{E}[\text{Number of renewals at } nd] \to \frac{d}{n} \quad , n \to \infty$$

REMARK 4.18. About the theorem.

(1) Even though the section was started with an example of a discrete renewal process, the part (2) of the Blackwell theorem applies to **any** non-lattice distribution. This includes any continuous distribution.
(2) All the parts of the theorem are true if $\mu = \infty$ $(1/\infty = 0)$.
(3) If $X_i > 0$, part (3) is $\Leftrightarrow \mathbf{P}\{\text{Renewal at } nd\} \to d/\mu$

PROOF. Not proven. □

Write for an infinitesimal increment $dy$:

$$m(dy) = \left( \underbrace{dm(y)}_{\text{Notation used sometimes}} \right) = m(y+dy) - m(y)$$
$$= \mathbb{E}[\text{Number of renewals in the interval } (y, y+dy]]$$

This is the *renewal measure.* The Blackwell renewal theorem says that:

$$m(dy) \simeq \frac{1}{\mu} dy.$$

LEMMA 4.19. *We have:*

$$m(dy) = \sum_{n=0}^{\infty} \mathbf{P}\left(S_n \in (y, y+dy]\right) \tag{4.5}$$

PROOF. The proof is straightforward (here we use a delay, therefore the sum starts from $n = 0$):

$$\begin{aligned}
m(dy) &= \mathbb{E}[N(y+dy) - N(y)] = \mathbb{E}[N(y+dy)] - \mathbb{E}[N(y)] = \\
&= \sum_{n=0}^{\infty} \mathbf{P}\left(N(y+dy) \geq n\right) - \sum_{n=0}^{\infty} \mathbf{P}\left(N(y) \geq n\right) \\
&= \sum_{n=0}^{\infty} \mathbf{P}\left(S_n \leq y+dy\right) - \sum_{n=0}^{\infty} \mathbf{P}\left(S_n \leq y\right) \\
&= \sum_{n=0}^{\infty} \mathbf{P}\left(S_n \in (y, y+dy]\right)
\end{aligned}$$

□

Many applications of the renewal theorem are concerned with the behavior of the process near a large time $t$. We need a final key of the puzzle before we proceed with the study of such applications and this key is provided in the next section.

## 4.3. The Key Renewal Theorem

This is the main result used in applications of the renewal processes. We will start with a definition.

DEFINITION 4.20 (Directly Riemann Integrable function). A function $h : [0, \infty) \to \mathbb{R}$ is called a Directly Riemann Integrable (DRI) function if the upper and lower mesh $\delta$ Darboux sums are finite and have the same limit as $\delta \to 0$.

*Reminder of lower (and upper) Darboux sum LDS (and UDS):*

Let $\pi = (t_0 = 0 < t_1 < t_2 < \dots)$ be a partition of $[0, \infty)$, with $max_i(t_i - t_{i-1}) \leq \delta$. Define:

$$LDS(h, \pi, \delta) = \sum_{n=1}^{\infty} \inf_{t \in [t_{n-1}, t_n]} h(t)(t_n - t_{n-1})$$

$$UDS(h, \pi, \delta) = \sum_{n=1}^{\infty} \sup_{t \in [t_{n-1}, t_n]} h(t)(t_n - t_{n-1})$$

EXAMPLE 4.21 (Example of Riemann integrable function which is **not** DRI). Let:

$$h(s) = \sum_{k=1}^{\infty} \mathbf{1}_{\{k \leq s < k + \frac{1}{k^2}\}}$$

Make a plot of this function to see what is happening. We have that:

$$\int_0^\infty h(s)ds = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{k=1}^\infty \frac{1}{k^2} < \infty,$$

so this function is Riemann integrable. However it is not DRI. Take the partition $\pi = (t_0 = 0, t_1 = \delta, t_2 = 2\delta, \ldots, t_n = n\delta, \ldots)$. Then:

$$\begin{aligned} UDS(h, \pi, \delta) &= \sum_{n=1}^\infty \sup_{t\in[(n-1)\delta, n\delta]} h(t)(n\delta - (n-1)\delta) \\ &= \delta \sum_{n=1}^\infty \sup_{t\in[(n-1)\delta, n\delta]} h(t) \end{aligned}$$

For any $\delta$ no matter how small but positive the last term is an infinite sum of 1's which is infinite.

PROPOSITION 4.22. *The following are sufficient conditions for a function to be DRI:*

(1) $h(t) \geq 0$, $\forall t > 0$
(2) $h$ *is nonincreasing*
(3) $\int_0^\infty h(t)dt < \infty$

PROOF. Not given. □

Now we are in the postion to be able to state the main theorem of this section.

THEOREM 4.23 (The Key Renewal Theorem). *For non-lattice* $X_1, X_2, \ldots$ *(any* $X_0$ *"delay" is fine) and if* $h$ *is a DRI function we have:*

$$\lim_{t\to\infty} \int_0^t h(t-y)m(dy) = \frac{1}{\mu}\int_0^\infty h(t)dt$$

PROOF. Skipped. □

This is a very powerful theorem. We shall see its application in the next section.

## 4.4. Applications of the Renewal Theorems

Refer back to Figure 1 on page 55. We can see there the current age at time $t$ and the remaining lifetime at $t$. Applications are concerned with these quantities when $t$ is large. So the question is: can we get distributions for these quantities? For example:

(a) $\mathbf{P}$(Age at time $t$ of the current item $> x) = \mathbf{P}(A(t) > x)$
(b) $\mathbf{P}$(Remaining lifetime of the item in use at $t > x) = \mathbf{P}(Y(t) > x)$

(c) $\mathbf{P}(\text{Total age of the item in use at } t > x) = \mathbf{P}(X_{N(t)+1} > x)$,

where we have use the obvious notations $A(t)$ to denote the age of the item in use at $t$ and $Y(t)$ to denote the residual life for the item in use at $t$.

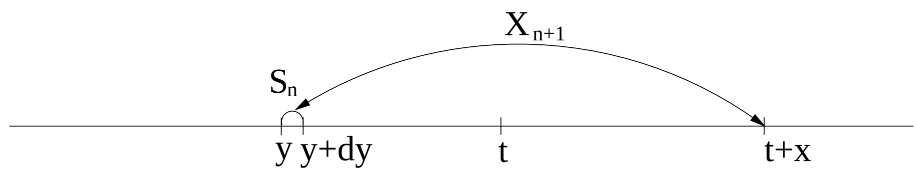


FIGURE 3. Deduction of the formula.

We will look at the process $Y(t)$ for exemplification (see Figure 3 on page 70). Recall that $X_0$ is the delay and we will use the convention $S_0 = X_0$. Note that this renewal is counted in the renewal process $N(t)$. We have:

$$\begin{aligned}\mathbf{P}(Y(t) > x) &= \mathbf{P}(S_{N(t)+1} - t > x) = \mathbf{P}(N(t) = 0, X_0 > t + x)\\ &\quad + \sum_{n=1}^{\infty} \int_0^t \mathbf{P}(N(t) = n, S_{n-1} \in (y, y + dy], X_n > t + x - y)\\ &= \mathbf{P}(X_0 > t + x) + \sum_{n=1}^{\infty} \int_0^t \mathbf{P}(S_{n-1} \in (y, y + dy], X_n > t + x - y)\\ &= (1 - F_0(t + x)) + \sum_{n=0}^{\infty} \int_0^t \mathbf{P}(S_n \in (y, y + dy])\mathbf{P}(X_{n+1} > t + x - y)\end{aligned}$$

Using the notation:

$$\overline{F}(x) = 1 - F(x)$$

we continue:

$$\begin{aligned}\mathbf{P}(Y(t) > x) &= \overline{F}_0(t + x) + \int_0^t \overline{F}(t + x - y) \sum_{n=0}^{\infty} \mathbf{P}(S_n \in (y, y + dy])\\ &= \overline{F}_0(t + x) + \int_0^t \overline{F}(t + x - y) m(dy)\\ &= \overline{F}_0(t + x) + \int_0^t h(t - y) m(dy),\end{aligned}$$

where we have used the Lemma 4.19 and we used the notation $h(s) = \overline{F}(s + x)$. Using now the fact that $\overline{F}_0(t + x) \xrightarrow{t\to\infty} 0$ (argue this yourselves) a direct application of the Key Renewal Theorem (KRT) 4.23 will yield:

$$\mathbf{P}(Y(t) > x) \xrightarrow{t\to\infty} \frac{1}{\mu}\int_0^\infty \overline{F}(s+x)ds = \frac{1}{\mu}\int_x^\infty \overline{F}(y)dy$$

This result is significant enough to make it a proposition.

Proposition 4.24. *Let $A(t)$ be the age at $t$ of the item and let $Y(t)$ be the residual life of the item alive at $t$. Then if $F$ is the c.d.f. of the lifetimes with mean lifetime $\mu$, then the distributions of $A(t)$ and $Y(t)$ for $t$ large have densities proportional with:*

$$f(y) = \frac{\overline{F}(y)}{\mu}$$

Proof. For $Y(t)$ the result is clear since from above we have:

$$\mathbf{P}(Y(t) \le x) \xrightarrow{t\to\infty}= \int_{-\infty}^x \frac{\overline{F}(y)}{\mu}dy$$

For $A(t)$ note that we have:

$$\{A(t) > x\} \Leftrightarrow \{Y(t-x) > x\} \quad \text{(No renewal in } [t-x,t]),$$

therefore

$$\lim_{t\to\infty}\mathbf{P}\{A(t) > x\} = \lim_{t\to\infty}\mathbf{P}\{Y(t-x) > x\} = \frac{1}{\mu}\int_x^\infty \overline{F}(y)dy$$

□

Remark 4.25. If the distribution of the delay has this special form: $\mathbf{P}(X_0 > x) = \frac{1}{\mu}\int_x^\infty \overline{F}(y)dy$ then $m(t) = \frac{t}{\mu}$ and the process is stationary (meaning that it looks the same regardless when you start observing it).

## 4.5. Special cases of Renewal Processes. Alternating Renewal process. Renewal Reward process.

**4.5.1. The Alternating Renewal process.** Let $\{(Z_n, Y_n)\}_{n=1}^\infty$ be i.i.d. pairs of random variables[5]. Note that the pairs for $i \neq j$ are independent but $Z_n$ and $Y_n$ can be dependent.

Let $X_n = Z_n + Y_n$. Let $S_n = \sum_{i=1}^n X_i$ which will give the renewal process.

*The story:* The $Z_i$'s represent the lightbulb lifetimes or the time that the system is ON, and the $Y_i$'s represent the replacement times or the time that the system is OFF.

[5]Here $(Z_1, Y_1)$ (delay) is allowed to have a different distribution than the rest

Denote the c.d.f of $Y_i$'s with $G$, the c.d.f of $Z_i$'s with $H$ and the c.d.f of $X_i$'s with $F$.

THEOREM 4.26. *If* $\mathbb{E}[X_n] < \infty$ *and* $F$ *is non-lattice we have:*

$$\mathbf{P}(\textit{The system is ON at time } t) \xrightarrow{t\to\infty} \frac{\mathbb{E}(Z_n)}{\mathbb{E}(X_n)} = \frac{\mathbb{E}(Z_n)}{\mathbb{E}(Z_n)+\mathbb{E}(Y_n)}$$

PROOF.

$$\begin{aligned}
\mathbf{P}(\text{ON at time } t) &= \mathbf{P}(Z_1 > t) + \sum_{n=0}^{\infty}\int_0^t \mathbf{P}(S_n \in (y, y+dy], Z_{n+1} > t-y) \\
&= \overline{H}_1(t) + \sum_{n=0}^{\infty}\int_0^t \mathbf{P}(S_n \in (y, y+dy])\mathbf{P}(Z_{n+1} > t-y) \\
&= \overline{H}_1(t) + \int_0^t \overline{H}(t-y)\sum_{n=0}^{\infty}\mathbf{P}(S_n \in (y, y+dy]) \\
&= \overline{H}_1(t) + \int_0^t \overline{H}(t-y)m(dy) \\
&\xrightarrow[\text{KRT}]{t\to\infty} \overline{H}_1(\infty) + \frac{1}{\mu}\int_0^{\infty}\overline{H}(t)dt
\end{aligned}$$

However, $\mathbb{E}[Z] = \int_0^\infty \mathbf{P}(Z > z)dz = \int_0^\infty \overline{H}(z)dz$ and $\mathbb{E}[X] = \mu$ so we are done. □

EXAMPLE 4.27. We have already seen that the distribution of $A(t)$ has density $\overline{F}(y)/\mu$. We will obtain this distribution again using the previous theorem about alternating renewal processes. Please read the next derivations since they provide examples of using this most useful theorem.

Once again we will deduce $\mathbf{P}(A(t) > x)$. Fix $x > 0$. Say that the system is ON during the first $x$ units of each lifetime and OFF the rest of that time. Mathematically, using the notation of the alternating renewal processes:

$$\begin{aligned}
Z_k &:= X_k \wedge x = \min(X_k, x) \\
Y_k &= X_k - Z_k
\end{aligned}$$

Then the theorem says:

$$\mathbf{P}(\text{System is ON at time t}) = \mathbf{P}(A(t) < x) \to \frac{\mathbb{E}(Z_n)}{\mu}$$

But we can calculate the limit since:

$$\begin{aligned}\mathbb{E}(Z_n) &= \int_0^\infty \mathbf{P}(Z_n > y)dy = \int_0^x \mathbf{P}(Z_n > y)dy + \int_x^\infty \mathbf{P}(Z_n > y)dy \\ &= \int_0^x \mathbf{P}(X_n > y)dy = \int_0^x \overline{F}(y)dy,\end{aligned}$$

which will give the density and finish the solution.

EXAMPLE 4.28 (Limiting distribution of the current lifetime $X_{N(t)+1}$). We want to calculate $\mathbf{P}(X_{N(t)+1} > x)$. Fix $x$. Construct an alternating renewal process using:

$$Z_n = X_n \mathbf{1}_{\{X_n > x\}}, \quad Y_n = X_n \mathbf{1}_{\{X_n \le x\}}$$

Then:

$$\mathbf{P}(\text{System is ON at time t}) = \mathbf{P}(X_{N(t)+1} > x) \to \frac{\mathbb{E}(Z_n)}{\mu}$$

Again we can calculate:

$$\begin{aligned}\mathbb{E}(Z_n) &= \int_0^\infty \mathbf{P}(Z_n > y)dy = \int_0^x \mathbf{P}(Z_n > y)dy + \int_x^\infty \mathbf{P}(Z_n > y)dy \\ &= \int_0^x \mathbf{P}(X_n > x)dy + \int_x^\infty \mathbf{P}(X_n > y)dy \\ &= x\mathbf{P}(X_n > x) + \int_0^\infty \overline{F}(y)dy \\ &= \int_0^\infty ydF(y) \quad \text{(Integrating by parts)}\end{aligned}$$

which will give the limiting distribution:

$$\mathbf{P}(X_{N(t)+1} > x) \to \frac{\int_0^\infty ydF(y)}{\mu}$$

Recall that if we denote $Y(t)$ the excess lifetime, we have already found its limiting distribution:

$$P(Y(t) > x) \to \frac{1}{\mu}\int_x^\infty \bar{F}(t)dy$$

We would like to find its expectation, or the limiting expected excess life, $\mathbb{E}[Y(t)]$. A first guess would be obviously the expectation of the previous distribution:

$$\mathbb{E}[Y(t)] = \int_0^\infty P(Y(t) > x)dx \to \frac{1}{\mu}\int_0^\infty \int_x^\infty \bar{F}(y)dydx$$

The guess turns out to be correct but we need to prove this.

PROPOSITION 4.29. *If $X$ is non-lattice with $\mathbb{E}[X^2] < \infty$, then*

$$\lim_{t\to\infty} \mathbb{E}[Y(t)] = \frac{\mathbb{E}[X^2]}{2\mu}$$

Note that one can show $\frac{\mathbb{E}[X^2]}{2\mu}$ and $\frac{1}{\mu}\int_0^\infty \int_x^\infty \bar{F}(y)dydx$ are the same quantities using a change the order of integration then integrating by parts.

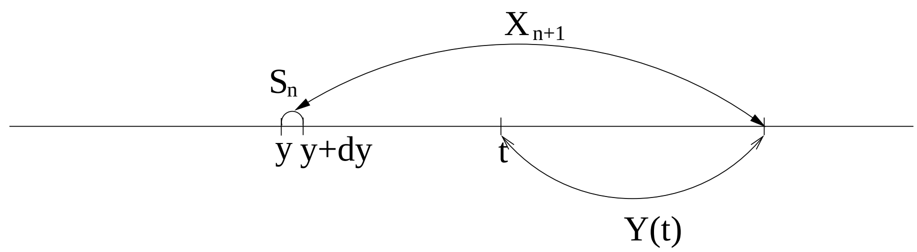


FIGURE 4. Relationship between $X_{n+1}$ and $Y(t)$

PROOF. We can go ahead and calculate:

$$\begin{aligned}
\mathbb{E}[Y(t)] &= \sum_{n=0}^{\infty} \mathbb{E}[Y(t)\mathbf{1}_{\{N(t)=n\}}] \\
&= \mathbb{E}[Y(t)\mathbf{1}_{\{N(t)=0\}}] + \sum_{n=1}^{\infty} \mathbb{E}[Y(t)\mathbf{1}_{\{N(t)=n\}}] \\
&= \mathbb{E}[(X_1 - t)\mathbf{1}_{\{X_1>t\}}] + \sum_{n=1}^{\infty} \int_0^t \mathbb{E}\left[ \underbrace{Y(t)}_{=X_{n+1}-(t-y)} \mathbf{1}_{\{S_n\in(y,y+dy],N(t)=n\}} \right] \\
&= \mathbb{E}[(X_1 - t)\mathbf{1}_{\{X_1>t\}}] + \sum_{n=1}^{\infty} \int_0^t \mathbb{E}\left[ (X_{n+1} - (t-y))\, \mathbf{1}_{\{S_n\in(y,y+dy]\}} \mathbf{1}_{\{X_{n+1}>t-y\}} \right] \\
&= \mathbb{E}[(X_1 - t)\mathbf{1}_{\{X_1>t\}}] + \int_0^t \mathbb{E}\left[ X - (t-y)\mathbf{1}_{\{X>t-y\}} \right] \underbrace{\sum_{n=1}^{\infty} \mathbb{E}\left[ \mathbf{1}_{\{S_n\in(y,y+dy]\}} \right]}_{=m(dy)} \\
&= \mathbb{E}[(X_1 - t)\mathbf{1}_{\{X_1>t\}}] + \int_0^t \mathbb{E}\left[ X - (t-y)\mathbf{1}_{\{X>t-y\}} \right] m(dy)
\end{aligned}$$

The first term in the above sum converges to 0 as $t \to \infty$ since $\mathbb{E}[X_1]$ is finite. We can write $h(t-y) = X - (t-y)\mathbf{1}_{\{X>t-y\}}$ and use the Key

Renewal Theorem for the second term. If we do that we obtain the limit as:

$$\begin{aligned}
\mathbb{E}[Y(t)] &\xrightarrow{t\to\infty} = \frac{1}{\mu}\int_0^\infty h(s)ds = \frac{1}{\mu}\int_0^\infty \mathbb{E}[(X-s)\mathbf{1}_{\{X>s\}}]ds \\
&= \frac{1}{\mu}\int_0^\infty \left[\int_s^\infty (x-s)dF(x)\right] ds \\
\text{Fubini} &= \frac{1}{\mu}\int_0^\infty \int_0^x (x-s)dsdF(x) \\
&= \frac{1}{\mu}\int_0^\infty -\frac{(x-s)^2}{2}\bigg|_0^x dF(x) \\
&= \frac{1}{\mu}\int_0^\infty \frac{x^2}{2}dF(x) = \frac{\mathbb{E}[X^2]}{2\mu}
\end{aligned}$$

□

COROLLARY 4.30. If $\mathbb{E}[X^2] < \infty$ and $F$ non-lattice then (for the un-delayed renewal process)

$$m(t) - \frac{t}{\mu} \xrightarrow{t\to\infty} \frac{\mathbb{E}[X^2]}{2\mu^2} - 1$$

PROOF. Note that we have shown that $\mathbb{E}[S_{N(t)+1}] = \mu \cdot (m(t)+1)$ However:

$$\mathbb{E}[t+Y(t)] = t + \mathbb{E}[Y(t)] \to t + \frac{\mathbb{E}[X^2]}{2\mu}$$

Since $S_{N(t)+1} = t + Y(t)$ we obtain:

$$m(t) + 1 \to \frac{t}{\mu} + \frac{\mathbb{E}[X^2]}{2\mu^2} \Rightarrow m(t) - \frac{t}{\mu} \to \frac{\mathbb{E}[X^2]}{2\mu^2} - 1$$

□

EXAMPLE 4.31. Let $X_1, X_2 \ldots$ iid $U[0,1]$. Then $\mu = \frac{1}{2}$, $\mathbb{E}[X^2] = \frac{1}{3}$

Then the corollary says for $t = 100$

$$\begin{aligned}
m(100) &\sim \frac{100}{m} + \frac{\mathbb{E}[X^2]}{2\mu^2} - 1 \\
&= \frac{100}{\frac{1}{2}} + \frac{\frac{1}{3}}{2\cdot(\frac{1}{2})^2} - 1 \quad \leftarrow \text{better approximation} \\
&= 199\frac{1}{3} \quad \text{(probably very accurate)}
\end{aligned}$$

**4.5.2. Renewal Reward Process.** Consider iid pairs: $(X_1, R_1), (X_2, R_2) \dots$

Story: At time $S_n = \sum_{i=1}^{n} X_i$ you get a reward $R_n$. Assume that $X_i \geq 0$, $\mathbb{E}[X_i] = \mu < \infty$, $\mathbb{E}[R_i] < \infty$.

Let $R_t = \sum_{i=1}^{N(t)} R_i$ the total reward up to time $t$

THEOREM 4.32. *Two results:*

(1)

$$\frac{R(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}[R]}{\mu} \qquad \text{as } t \to \infty$$

(2)

$$\frac{\mathbb{E}[R(t)]}{t} \to \frac{\mathbb{E}[R]}{\mu} \qquad \text{as } t \to \infty$$

PROOF. **Part (1):** We have:

$$\frac{R(t)}{t} = \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \, \frac{N(t)}{t}$$

The first term in the product above converges to the $\mathbb{E}[R]$ using the strong law of large numbers and the second term converges to $1/\mu$ by the renewal SLLN. Therefore, we get the result in part (1).

**Part (2):** We have:

Using Wald for $N(t) + 1$ which is a stopping time,

$$\begin{aligned} \mathbb{E}[R(t)] &= \mathbb{E}[\sum_{i=1}^{N(t)} R_i] = \mathbb{E}[\sum_{i=1}^{N(t)+1} R_i] - \mathbb{E}[R_{N(t)+1}] \\ &= \mathbb{E}[N(t)+1]\mathbb{E}[R_n] - \mathbb{E}[R_{N(t)+1}] = (m(t)+1)\,\mathbb{E}(R) - \mathbb{E}[R_{N(t)+1}] \end{aligned}$$

This implies dividing with $t$ and taking the limit as $t \to \infty$:

$$\frac{\mathbb{E}[R(t)]}{t} = \frac{(m(t)+1)}{t}\mathbb{E}(R) - \frac{\mathbb{E}[R_{N(t)+1}]}{t} \xrightarrow{t\to\infty} \frac{\mathbb{E}(R)}{\mu} - \lim_{t\to\infty} \frac{\mathbb{E}[R_{N(t)+1}]}{t},$$

where we used the elementary renewal theorem for the first term. To complete the proof we have to show that $\lim_{t\to\infty} \mathbb{E}[R_{N(t)+1}]/t = 0$. We have:

$$
\begin{aligned}
\mathbb{E}[R_{N(t)+1}] &= \mathbb{E}[R_1 \mathbf{1}_{\{X_1>t\}}] + \sum_{n=1}^{\infty} \int_0^t \mathbb{E}\left[R_{n+1}\mathbf{1}_{\{X_{n+1}>t-y, S_n\in(y,y+dy], N(t)=n\}}\right] \\
&= \mathbb{E}[R_1 \mathbf{1}_{\{X_1>t\}}] + \int_0^t \sum_{n=1}^{\infty} \mathbb{E}\left[R_{n+1}\mathbf{1}_{\{X_{n+1}>t-y\}}\right] \mathbb{E}\left[\mathbf{1}_{\{S_n\in(y,y+dy]\}}\right] \\
&= \mathbb{E}[R_1 \mathbf{1}_{\{X_1>t\}}] + \int_0^t \mathbb{E}\left[R_2\mathbf{1}_{\{X_2>t-y\}}\right] \sum_{n=1}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{S_n\in(y,y+dy]\}}\right] \\
&= \mathbb{E}[R_1 \mathbf{1}_{\{X_1>t\}}] + \int_0^t h(t-y)m(dy),
\end{aligned}
$$

where we denoted $h(t-y) = \mathbb{E}\left[R_2\mathbf{1}_{\{X_2>t-y\}}\right]$, to apply the KRT. The first term converges to 0 as $t \to \infty$ (justify), and we obtain the limit:

$$
\lim_{t\to\infty} \mathbb{E}[R_{N(t)+1}] = \frac{1}{\mu}\int_0^{\infty} h(t)dt = \frac{1}{\mu}\int_0^{\infty} \mathbb{E}\left[R_2\mathbf{1}_{\{X_2>t\}}\right] < \frac{\mathbb{E}(R)}{\mu} < \infty
$$

Thus, dividing with $t$ and taking the limit we obtain 0, which finishes the proof. □

## 4.6. The Renewal Equation. Convolutions.

Often the quantity of interest in renewal theory $Z(t)$ satisfies an equation of the form:

$$
Z(t) = z(t) + \int_0^t Z(t-y)F(dy)
$$

where $F(t)$ = c.d.f. of interarriaval time, and $z(t)$ is the some known function with the properties:

- $z(t) = 0$ if $t < 0$
- $z$ bounded on finite interval

An equation of this type is called a **renewal equation**

Example 4.33. $m(t)$ satisfies:

$$
m(t) = F(t) + \int_0^t m(t-y)F(dy)
$$

Example 4.34. $P(Y(t) > x)$:

$$
P(Y(t) > x) = \bar{F}(t+x) + \int_0^t P(Y(t-y) > x)F(dy)
$$

EXAMPLE 4.35. $\mathbb{E}[Y(t)]$:

$$\mathbb{E}[Y(t)] = \mathbb{E}[X_1 - t]I_{\{X_1>t\}} + \int_0^t \mathbb{E}[Y(t-y)]dF(y)$$

The next theorem will provide a way to solve the renewal equation.

THEOREM 4.36. *If $F(0-) = 0$, $F(0) < 1$, $z(t)$ is bounded on finite intervals and $z(t) = 0$ for $t < 0$ then the renewal equation*

$$Z(t) = z(t) + \int_0^t Z(t-s)dF(s)$$

*has a unique solution, bounded on finite intervals given by*

$$Z(t) = z(t) * m_0(t) = \int_0^t z(t-y)m_0(dy) = \sum_{n=0}^{\infty} \int_0^t z(t-y)dF_n(y)$$

*where*

$$m_0(t) = \sum_{n=0}^{\infty} F_n(t) = \sum_{n=0}^{\infty} P(S_n \le t)$$
$$F_n(t) = \underbrace{F * F \ldots * F}_{n \text{ times}}, \qquad \text{with } S_0 = 0$$

**Properties of Convolution** Let $F$, $G$ c.d.f.'s with $F(0-) = G(0-) = 0$, $z$ as in the theorem. Then:

(1) $F * G = G * F$
(2) z*(F*G)=(z*F)*G
(3) z*(F+G)=z*F+z*G
(4) If $G$ has density $g$ then $F * G$ has density $g * F = \int_0^t g(t-y)F(dy)$

PROOF OF THE THEOREM ON RENEWAL EQUATION. **Part 1.** Existence of the solution.

$$
\begin{aligned}
z * m_0(t) &= \sum_{n=0}^{\infty} z * F_n(t) \\
&= z * F_0(t) + \sum_{n=1}^{\infty} z * F_n(t) \\
&= z(t) * F_0 + \left[\sum_{n=0}^{\infty} z * F_n(t)\right] * F(t) \\
&= z(t) + (z * m_0) * F(t) \\
&= z(t) + \int_0^t (z * m_0)(t-s)dF(s)
\end{aligned}
$$

Note that we used the fact $F_0(t) = P(S_0 \le t) = \mathbf{1}_{\{t \ge 0\}}$

This shows that $z * m_0$ is a solution for the renewal equation.

**Part 2.** Uniqueness:

Assume that there exist $Z_1(t)$ and $Z_2(t)$ 2 solutions of the renewal equation. Let $V(t) = (Z_1 - Z_2)(t)$. By definition $V(t)$ should also solve the renewal equation, i.e.,

$$
\begin{aligned}
V(t) &= (Z_1 - Z_2)(t) \\
&= z(t) + \int_0^t Z_1(t-s)dF(s) - z(t) - \int_0^t Z_2(t-s)dF(s) \\
&= \int_0^t V(t-s)dF(s) = V * F(t)
\end{aligned}
$$

Repeat the argument:

$$V(t) = V * F(t) = V * F_2(t) = \ldots = V * F_k(t), \quad \forall k$$

which implies:

$$
\begin{aligned}
V(t) &= \int_0^t V(t-y)F_k(dy) \\
&\le \sup_{0 \le s \le t} V(s) \int_0^t dF_k(s) \\
&= \sup_{0 \le s \le t} V(s) F_k(t) \xrightarrow{k \to \infty} 0
\end{aligned}
$$

Because $F_k(t) = P(X_1 + X_2 + \ldots + X_k \le t) \xrightarrow{k \to \infty} 0$, $\forall t$ fixed. (CLT or SLLN) $\square$

THEOREM 4.37. *(true for both lattice and non-lattice case) If $X_1$ has distribution*

$$P(X_1 > x) = \int_0^\infty \frac{1}{\mu}\overline{F}(y)dy \stackrel{def}{=} F_e(x)$$

*This is called the equilibrium distribution; the process with the delay $X_1$ having this distribution is called the equilibrium renewal process. Let*

$$m_D(t) \stackrel{def}{=} \sum_{n=1}^{\infty} P(S_n \le t) = \sum_{n=0}^{\infty} F_e * F_n(t),$$

*and $Y_D(t)$ be the residual lifetime at $t$ for the delayed process. Then:*

(1) $m_D(t) = \frac{t}{\mu}$
(2) $P(Y_D(t) > x) = \overline{F}_e(x)$ *for all* $t > 0$
(3) $\{N_D(t)\}_t$ *has stationary increments.*

PROOF. **Part 1.**

$$\begin{aligned} m_D(t) &= F_e(t) + \left(F_e * \sum_{n=1}^{\infty} F_n\right)(t) \\ &= F_e(t) + \left(F_e * \sum_{n=0}^{\infty} F_n\right) * F(t) \\ &= F_e(t) + m_D(t) * F(t) \end{aligned}$$

which implies that $m_D(t)$ solves a renewal equation with $z(t) = F_e(t)$

If we show that $\frac{t}{\mu}$ also solves the renewal equation with the same $z(t)$, we are done.

Check yourself that $h(t) = \frac{t}{\mu}\mathbf{1}_{\{t>0\}}$ also solves the same renewal equation. By uniqueness of the solution we are done.

**Part 2**We have using the usual renewal argument:

$$
\begin{aligned}
P(Y_D(t) > x) &= \mathbf{P}(X_1 > t + x) + \int_0^t \overline{F}(t - y + x) m_D(dy) \\
(\text{From (i)} \Longrightarrow) &= \overline{F}_e(t + x) + \int_0^t \overline{F}(t - y + x) \frac{dy}{\mu} \\
&= \int_{t+x}^{\infty} \frac{1}{\mu} \overline{F}(y) dy + \int_0^t \frac{1}{\mu} \overline{F}(t - y + x) dy \\
(\text{c.v. } v = t - y + x) &= \int_{t+x}^{\infty} \frac{1}{\mu} \overline{F}(y) dy - \int_{t+x}^{x} \frac{1}{\mu} \overline{F}(v) dv \\
&= \int_{x}^{\infty} \frac{1}{\mu} \overline{F}(y) dy \\
&= \overline{F}_e(x) \quad \Longrightarrow \text{ DONE.}
\end{aligned}
$$

**Part 3.** This part follows from part (2) using the fact that $N_D(t + s) - N_D(s)$ is the number of renewals in a time interval length $t$ of a delayed renewal process. $\square$

CHAPTER 5

# Special Chapter about some applications of the notions learned thus far.

At this point we are going to stop and look to some interesting examples of discrete processes.

## 5.1. Random Walk on integers in $\mathbb{R}^d$

Let $\vec{X}_k = (X_k^{(1)}, X_k^{(2)} \dots X_k^{(d)}) \in \mathbb{R}^d$ a random vector. Each $X_k^{(i)}$ is independent of the others and it is $\pm 1$ each with probability $\frac{1}{2}$.

$$\text{Then } \vec{S}_n = \sum_{k=1}^{n} \vec{X}_k \quad \text{is a } d\text{-dimensional random walk}$$

Remark 5.1. The sum above is done componentwise, it is not the regular summation of the vectors. I use the notation $\vec{X}$ to symbolize the fact that $X$ has more than one dimensions nothing more, there is no origin, directional angle or size involved in the notation.

We will talk next about some common questions regularly asked about this process.

**Question:** Once started from $(0, 0, \dots, 0)$, would the process come back to $\vec{0}$? OR is $S_n = (0, 0, \dots, 0)$ for some $n$?

Answer: For $n$ odd, $P(S_n = \vec{0}) = 0$. For $n$ even say equal to $2k$

$$\begin{aligned} P(\vec{S}_{2k} = \vec{0}) &= P\left(S_{2k}^{(1)} = 0, S_{2k}^{(2)} = 0, \dots S_{2k}^{(d)} = 0\right) \\ &= \left[P\left(S_{2k}^{(i)} = 0\right)\right]^d \end{aligned}$$

$\square$

Claim: $\left[P\left(S_{2k}^{(i)} = 0\right)\right]^d = \left(\frac{\text{constant}}{\sqrt{k}}\right)^d$

Why?

Note that we have a total of $2k$ steps and $S_{2k}^{(i)}$ is now 1-dimensional. To get back to 0 once you start from it you need $k$ steps up (values

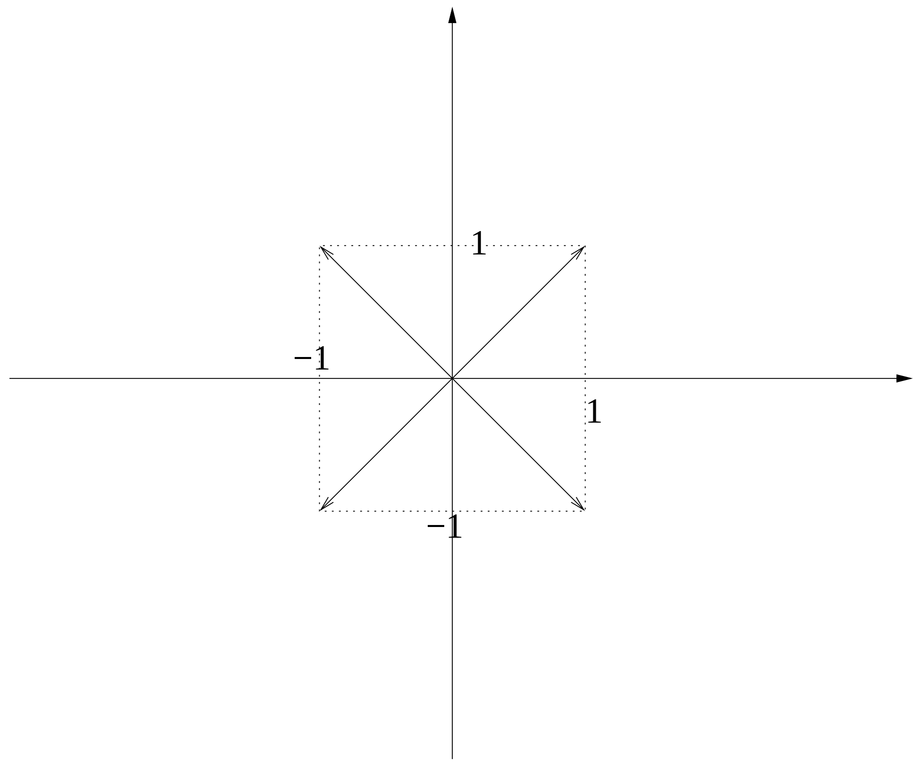


FIGURE 1. Random walk in 2-dimensions. Possible values for the first jump each with probability 1/4

of 1) and $k$ steps down (values of $-1$). But then the number of such paths is:

$$2k \text{ total steps} \Rightarrow \begin{cases} k \text{ forward steps} \\ k \text{ back steps} \end{cases} \rightarrow \binom{2k}{k}$$

Probability of any such path is $(\frac{1}{2})^{2k}$, which implies

$$P\left(S_{2k}^{(i)} = 0\right) = \binom{2k}{k}\left(\frac{1}{2}\right)^{2k}$$

Now use Stirling's formula for the combinatorial term, (i.e., $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$):

$$\begin{aligned}
P\left(S_{2k}^{(i)} = 0\right) &= \frac{(2k)!}{(k!)^2}\left(\frac{1}{2}\right)^{2k} \\
&= \frac{\sqrt{2\pi 2k}(2k)^{2k}e^{-2k}}{(\sqrt{2\pi k}k^k e^{-k})^2}\left(\frac{1}{2}\right)^{2k} \\
&= \frac{2\sqrt{\pi k}2^{2k}(k^{2k}e^{-2k})}{2\pi k(k^{2k}e^{-2k})}\left(\frac{1}{2}\right)^{2k} \\
&= \frac{1}{\sqrt{\pi k}} = \frac{\text{constant}}{\sqrt{k}}
\end{aligned}$$

which proves the claim.

THEOREM 5.2 (Polya). *If $d = 1, 2$ then you come back to $\vec{0}$ infinitely often. If $d \geq 3$ eventually you never come back to $\vec{0}$.*

PROOF. Let $p_d = \mathbf{P}\{\text{You never come back to } \vec{0} \textbf{ at all} \text{ in } \mathbb{R}^d\}$. Then:

$$\sum_{n=1}^{\infty} \mathbf{1}_{\{\vec{S}_n = \vec{0}\}} = \{\text{number of times you return to } \vec{0}\}$$

then $\sum_{n=1}^{\infty} \mathbf{1}_{\{\vec{S}_n=\vec{0}\}}$ is a Geometric($p_d$) (number of failures) random variable. This is clear if you consider coming back to $\vec{0}$ a failure and not coming back a success. Therefore we can write:

$$\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbf{1}_{\{\vec{S}_n=\vec{0}\}}\right] = \frac{1}{p_d} - 1$$

Using Fubini and the previous claim:

$$\begin{aligned}
\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbf{1}_{\{\vec{S}_n=\vec{0}\}}\right] &= \sum_{n=1}^{\infty}\left(\mathbb{E}[\mathbf{1}_{\{\vec{S}_n=\vec{0}\}}]\right) = \sum_{n=1}^{\infty} P\left(\vec{S}_n = \vec{0}\right) = \sum_{\substack{n=2k \\ k=1}}^{\infty} \frac{c^d}{k^{\frac{d}{2}}} \\
&= c^d \sum_{k=1}^{\infty} \frac{1}{k^{\frac{d}{2}}}
\end{aligned}$$

Therefore:

$$\frac{1-p_d}{p_d} = \begin{cases} < \infty & \text{if } d \leq 2 \\ \infty & \text{if } d > 2 \end{cases}$$

which implies

$$p_d > 0 \Leftrightarrow d \geq 3$$
$$p_d = 0 \Leftrightarrow d = 1, 2$$

As a conclusion when $d = 1, 2$ the number of visits to 0 is $\infty$. When $d >= 3$ the number of visits is finite a.s. which means that eventually you will drift to infinity. Moral: Do not get drunk while driving a spaceship. $\square$

REMARK 5.3. Considering a renewal event the event that the random walk returns to the origin $\vec{0}$ we can easily see that the random walk produces a renewal process.

## 5.2. Age dependent Branching processes

Story: Let $F$ to be the lifetime distribution, with $F(0) = 0$, $P_j$ to be the probability that at death we get exactly $j$ offsprings, $j = 0, 1, 2 \ldots$.
Each offspring then acts independently of others and produce their own offspring according to $P_j$, and so on and so forth.

Let $X(t)$ denote the number of organisms alive at time $t$. $\{X\}_{t>0}$ is called an age dependent branching process with $X(0) = 1$.

**Quantity of interest :** $M(t) = \mathbb{E}[X(t)]$

$$m = \sum_{j=0}^{\infty} jP_j = \text{number of offsprings (assumed to be } > 1)$$

**Special case:** If lifetimes are identically equal to 1, then

$$M(k) = m^k$$

In this particular situation $X(k)$ is called a Galton-Watson process (which is also a Markov Chain, as we will see later). This process was invented in 1873 by the people whose name it bears as result of a study initiated at the request of the crown to see if the aristocratic surnames were dying out in England of that time.

**Remark :** usually

$$\frac{X(k)}{m^k} \to Z$$

with $Z$ a random variable finite a.s..

THEOREM 5.4. *If $X(0) = 1$ and $F$ is non-lattice then*

$$e^{-\alpha t} M(t) \xrightarrow{t \to \infty} \frac{m-1}{m^2 \alpha \int_0^{\infty} x e^{-\alpha x} dF(x)}$$

*where $\alpha > 0$ is the solution of the equation:* $\int_0^\infty e^{-\alpha t} dF(t) = \frac{1}{m}$

REMARK 5.5. The theorem simply says that $M(t) \sim \text{constant} \cdot e^{-\alpha t}$

PROOF. Using $ds = (s, s+ds]$ and the renewal argument:

$$\begin{aligned}
M(t) &= \mathbb{E}[X(t)] \\
&= \mathbb{E}[X(t)\mathbf{1}_{\{\text{1st life } > t\}}] + \mathbb{E}[X(t)\mathbf{1}_{\{\text{1st life} \le t\}}] \\
&= \mathbb{E}[1\mathbf{1}_{\{T_1 > t\}}] + \int_0^t \mathbb{E}[\underbrace{N}_{\text{Nr. of offsprings}} X(t-s)\mathbf{1}_{\{T_1 \in ds\}}] \\
&= P(T_1 > t) + \int_0^t m\mathbb{E}[X(t-s)\mathbf{1}_{\{T_1 \in ds\}}] \qquad \text{, note } m = \mathbb{E}[N] \\
&= \overline{F}(t) + \int_0^t mM(t-s)dF(s)
\end{aligned}$$

This looks a lot like a renewal equation except for the $m$. To eliminate it multiply both sides by $e^{-\alpha t}$

$$\begin{aligned}
\Rightarrow M(t)e^{-\alpha t} &= \overline{F}(t)e^{-\alpha t} + \int_0^t e^{-\alpha t} mM(t-s)dF(s) \\
&= \overline{F}(t)e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} M(t-s)\underbrace{me^{-\alpha s}dF(s)}_{=dG(s)}
\end{aligned}$$

We denoted $dG(s) = me^{-\alpha s}dF(s)$, OR $G(t) = \int_0^t me^{-\alpha s}dF(s)$

$G$ is a c.d.f. because $\alpha$ is a solution of $\int_0^\infty e^{-\alpha t}dF(t) = \frac{1}{m}$. Its definition implies that $G(0-) = 0$.

Thus, we obtain a renewal equation $Z(t) = z(t) + \int_0^t Z(t-s)dG(s)$, where:

$$\begin{aligned}
Z(t) &= e^{-\alpha t}M(t) \\
z(t) &= e^{-\alpha t}\overline{F}(t)
\end{aligned}$$

Recall that the unique solution is

$$\begin{aligned}
Z(t) = z * m_0(t) &= \int_0^t z(t-s)m_0(ds) \qquad \left(\text{with } m_0 \text{ given by } \sum_{n=0}^\infty G_n(s)\right) \\
&\xrightarrow{\text{KRT}} \frac{1}{\mu_G}\int_0^\infty z(t)dt,
\end{aligned}$$

provided that we can apply the Key Renewal Theorem. Looking back we see that $Z(t) \to \frac{1}{\mu_G}\int_0^\infty e^{-\alpha t}\overline{F}(t)dt$. Now let us calculate the limit, while at the same time showing that $z(t)$ is DRI. Using that $\overline{F}(t) = \int_t^\infty dF(x)$ we have:

$$\begin{aligned}
\int_0^\infty z(t)dt &= \int_0^\infty \left(\int_t^\infty dF(x)\right) e^{-\alpha t}dt \\
&= \int_0^\infty \left(\int_0^x e^{\alpha t}dt\right) dF(x) \qquad \text{(Fubini)} \\
&= \int_0^\infty \frac{1}{\alpha}(1-e^{-\alpha x})dF(x) \\
&= \frac{1}{\alpha}\left[\underbrace{\int_0^\infty dF(x)}_{=1} - \underbrace{\int_0^\infty e^{-\alpha x}dF(x)}_{=\frac{1}{m}\text{ (def. of }\alpha)}\right] \\
&= \frac{1}{\alpha}\left[1-\frac{1}{m}\right] < \infty
\end{aligned}$$

Thus, $z(t)$ is Riemann integrable, also it is decreasing and positive therefore it is DRI. All that remains is to calculate $\mu_G$.

$$\mu_G = \int_0^\infty x dG(x) = \int_0^\infty x m e^{-\alpha x} dF(x) = m\int_0^\infty x e^{-\alpha x} dF(x)$$

Hence,

$$Z(t) = e^{-\alpha t}M(t) \to \frac{\frac{1}{\alpha}\left(1-\frac{1}{m}\right)}{m\int_0^\infty \alpha e^{-\alpha x}dF(x)}$$

And a little algebra shows that this is exactly the formula we need to prove. $\square$

Remark 5.6. What if $m < 1$? If $\exists \alpha < 0$ with $\int_0^\infty e^{-\alpha x}dF(x) = \frac{1}{m}$ and $e^{-\alpha x}\overline{F}(x)$ is DRI, then the same result is true.

In either case $m > 1$, $m < 1$; $\mu_G = \infty$ is possible and it will not change the answers.

Exercise 22. **Question** For a Branching Process what is the probability that $X(t) = 0$ eventually? (population dies out).
Think about this.
Guesses.

For $m < 1$, it is kind of obvious that $P(\text{Population dies out}) = 1$.

If $m = 1$, then $P(\text{Population dies out}) = 1$ except when the number of offsprings is exactly 1.

What if $m > 1$? $P(\text{Population dies out}) > 0$ iff $P(0 \text{ offsprings}) > 0$