Homework 1

Ma635 Real Analysis I

due Thursday Sep 25, 2006

1) *(Bernoulli Inequality).* Given $a \in \mathbb{R}$, $a > -1$, $a \neq 0$, show that $(1 + a)^n > 1 + an$, for all $n \in \mathbb{N}^*$. (*Hint.* Induction)

2) *(Bolzano- Weierstrass Theorem).* Every bounded sequence of real numbers has a convergent subsequence. (*Hint.* There are many ways to prove this, for example use $\limsup$ arguments or show that every sequence of real numbers has a monotone subsequence).

3) Show that a sequence of real numbers is convergent if and only if it is Cauchy. (*Hint.* Once again use $\limsup$, $\liminf$ arguments)

4) The Hilbert cube H^∞ is the collection of all real sequences $x = x_n{}_{n=1}^{\infty}$, with $|x_n| \leq 1$, $\forall n \in \mathbb{Z}$.

   (i) Show that
   $$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n - y_n|$$
   defines a metric on H^∞.

   (ii) Let $x, y \in H^\infty$, $k \in \mathbb{N}$. Let $M_k = \max\{|x_1 - y_1|, \dots, |x_k - y_k|\}$. Show that
   $$\frac{1}{2^k} M_k \leq d(x, y) \leq M_k + \frac{1}{2^k}$$

5) Let $\mathscr{C}[a, b]$ be the space (collection) of all continuous real valued functions defined on the closed interval $[a, b]$.

(i) Let
$$d(f,g) = \max_{a \le t \le b} |f(t) - g(t)| \quad , \forall f,g \in \mathscr{C}[a,b].$$
Show that $d(\cdot,\cdot)$ is a metric (called the Fréchet metric).

(ii) For the space $\mathscr{C}[0,1]$ show that the following:
$$\rho(f,g) = \int_0^1 |f(t) - g(t)| dt$$
$$\sigma(f,g) = \int_0^1 \min\{|f(t) - g(t)|, 1\} dt$$
are also metrics.

6) (i) Consider the sequence $\left\{\frac{1}{n}\right\}_{n \ge 1}$ living in the space $\mathscr{X} = (0,1]$. The sequence is Cauchy but it is not convergent. What does this example violate and why?

(ii) Let a space $\mathscr{X}$ be discrete (countable number of elements) and a distance defined on it. Let $\{x_n\}$ be a Cauchy sequence in this space. Then this Cauchy sequence is convergent. Why?

7) Select one of the bold statements and argument your choice.

(i) Any finite set is **open/closed/neither**.

(ii) The interval of the type $(-\infty, a]$ is **open/closed/neither**.

(iii) In a discrete space every subset is **open/closed/neither**.

(iv) $(0,1]$ is **open/closed/neither** in $\mathbb{R}$.

(v) $(0,1]$ is relatively **open/closed/neither** w.r.t. $(0,\infty)$.

(vi) $(0,1]$ is relatively **open/closed/neither** w.r.t. $(-\infty,1)$.