# Homework 2

## *Ma635 Real Analysis I*

due Thursday Oct 11, 2006

1) Let $f : (\mathcal{X}, d) \to (\mathcal{Y}, \rho)$ be one-to-one and onto. Prove that f is a homeomorphism if and only if $f(\bar{A}) = \overline{f(A)}$ for every subset $A$ of $\mathcal{X}$.

2) Let $[a, b]$ be any closed, bounded interval in $\mathbb{R}$, and let $\sigma : [0, 1] \to [a, b]$ be defined as $\sigma(t) = a + t(b - a)$. Prove that:

   (i) $\sigma$ is a homeomorphism.

   (ii) $f \in \mathscr{C}[a, b]$ if and only if $f \circ \sigma \in \mathscr{C}[0, 1]$.

   (iii) The map $f \mapsto f \circ \sigma$ is an isometry from $\mathscr{C}[a, b]$ to $\mathscr{C}[0, 1]$.

   The map $T(f) = f \circ \sigma$ actually does much more; it is both an algebra and a lattice isomorphism (i.e., it also preserves the algebraic and order structures). Specifically, given any $f, g \in \mathscr{C}[a, b]$ check that:

   (iv) $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$ for all $\alpha, \beta \in \mathbb{R}$.

   (v) $T(fg) = T(f)T(g)$.

   (vi) $T(f) \leq T(g)$ if and only if $f \leq g$. Thus for all practical purposes $\mathscr{C}[a, b]$ is identical with $\mathscr{C}[0, 1]$.

3) Last time you showed that on $\mathscr{C}[a, b]$ the following is a metric (Fréchet metric) $d(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|$. Also due to Fréchet, consider $\mathscr{C}(\mathbb{R})$ (that is continuous functions defined on $\mathbb{R}$) and the following definition:

$$d_n(f, g) = \max_{|t| \leq n} |f(t) - g(t)| \quad , \forall f, g \in \mathscr{C}(\mathbb{R}).$$

Show that the $d_n(\cdot,\cdot)$ is a pseudo-metric (that is it has the properties of a metric – finiteness, symmetry and triangle inequality, minus the one that says $d(x,y) = 0$ if and only if $x = y$). A pseudo-metric will allow distinct points to be 0 distance apart.

Then show that:

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f,g)}{1 + d_n(f,g)} \quad , \forall f, g \in \mathscr{C}(\mathbb{R}),$$

defines a proper metric on $\mathscr{C}(\mathbb{R})$.

4) Do problem 1 on page 31.

5) Do problem 8 on page 31.