Section 2.4 problems:

Problem 4 solution:

First we note that $M(x, y) = \sqrt{-2y - y^2}$ depends only on y and $N(x, y) = 3 + 2x - x^2$ depends only on x. So, the equation is separable. It is not linear with x as independent variable because $M(x, y)$ is not a linear function of y. Similarly, it is not linear with y as independent variable because $N(x, y)$ is not a linear function of x. Computing

$$\frac{\partial M}{\partial y} = \frac{1}{2}\left(-2y - y^2\right)^{-1/2}(-2 - 2y) = -\frac{1+y}{\sqrt{-2y - y^2}},$$
$$\frac{\partial N}{\partial x} = 2 - 2x,$$

we see that the equation (5) in Theorem 2 is not satisfied. Therefore, the equation is not exact.

Problem 6 solution:

It is separable, linear with x as independent variable, and not exact because

$$\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = 0.$$

Problem 8 solution:

Here, $M(x, y) = 2x + y\cos(xy)$, $N(x, y) = x\cos(xy) - 2y$. Since $M(x, y)/N(x, y)$ cannot be expressed as a product $f(x)g(y)$, the equation is not separable. We also conclude that it is not linear because $M(x, y)/N(x, y)$ is not a linear function of y and $N(x, y)/M(x, y)$ is not a linear function of x. Taking partial derivatives

$$\frac{\partial M}{\partial y} = \cos(xy) - xy\sin(xy) = \frac{\partial N}{\partial x},$$

we see that the equation is exact.

Problem 16 solution:

Computing

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(ye^{xy} - y^{-1}\right) = e^{xy} + xye^{xy} + y^{-2},$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(xe^{xy} + xy^{-2}\right) = e^{xy} + xye^{xy} + y^{-2},$$

we see that the equation is exact. Therefore,

$$F(x, y) = \int \left(ye^{xy} - y^{-1}\right) dx = e^{xy} - xy^{-1} + g(y).$$

So,

$$\frac{\partial F}{\partial y} = xe^{xy} + xy^{-2} + g'(y) = N(x, y) \qquad \Rightarrow \qquad g'(y) = 0.$$

Thus, $g(y) = 0$, and the answer is $e^{xy} - xy^{-1} = C$.

Problem 20 solution:

We find

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[y\cos(xy)\right] = \cos(xy) - xy\sin(xy),$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[x\cos(xy)\right] = \cos(xy) - xy\sin(xy).$$

Therefore, the equation is exact and

$$F(x, y) = \int \left(x\cos(xy) - y^{-1/3}\right) dy = \sin(xy) - \frac{3}{2}y^{2/3} + h(x)$$
$$\frac{\partial F}{\partial x} = y\cos(xy) + h'(x) = \frac{2}{\sqrt{1 - x^2}} + y\cos(xy)$$
$$\Rightarrow \qquad h'(x) = \frac{2}{\sqrt{1 - x^2}} \qquad \Rightarrow \qquad h(x) = 2\arcsin x,$$

and a general solution is given by

$$\sin(xy) - \frac{3}{2}y^{2/3} + 2\arcsin x = C.$$

Problem 26 solution:

Taking partial derivatives M_y and N_x, we find that the equation is exact. So,

$$F(x,y) = \int (\tan y - 2)\, dx = x(\tan y - 2) + g(y),$$
$$\frac{\partial F}{\partial y} = x\sec^2 y + g'(y) = x\sec^2 y + y^{-1}$$
$$\Rightarrow \qquad g'(y) = y^{-1} \qquad \Rightarrow \qquad g(y) = \ln|y|,$$

and

$$x(\tan y - 2) + \ln|y| = C$$

is a general solution. Substituting $y(0) = 1$ yields $C = 0$. Therefore, the answer is

$$x(\tan y - 2) + \ln y = 0.$$

(We removed the absolute value sign in the logarithmic function because $y(0) > 0$.)

Problem 30 solution:

(a) Differentiating, we find that

$$\frac{\partial M}{\partial y} = 5x^2 + 12x^3 y + 8xy\,,$$
$$\frac{\partial N}{\partial x} = 6x^2 + 12x^3 y + 6xy\,.$$

Since $M_y \neq N_x$, the equation is not exact.

(b) Multiplying given equation by $x^n y^m$ and taking partial derivatives of new coefficients yields

$$\frac{d}{dy}\left(5x^{n+2}y^{m+1} + 6x^{n+3}y^{m+2} + 4x^{n+1}y^{m+2}\right)$$
$$= 5(m+1)x^{n+2}y^m + 6(m+2)x^{n+3}y^{m+1} + 4(m+2)x^{n+1}y^{m+1}$$
$$\frac{d}{dx}\left(2x^{n+3}y^m + 3x^{n+4}y^{m+1} + 3x^{n+2}y^{m+1}\right)$$
$$= 2(n+3)x^{n+2}y^m + 3(n+4)x^{n+3}y^{m+1} + 3(n+2)x^{n+1}y^{m+1}.$$

In order that these polynomials are equal, we must have equal coefficients at similar monomials. Thus, n and m must satisfy the system

$$\begin{cases} 5(m+1) = 2(n+3) \\ 6(m+2) = 3(n+4) \\ 4(m+2) = 3(n+2). \end{cases}$$

Solving, we obtain $n = 2$ and $m = 1$. Therefore, multiplying the given equation by x^2y yields an exact equation.

(c) We find

$$\begin{aligned} F(x,y) &= \int \left(5x^4y^2 + 6x^5y^3 + 4x^3y^3\right) dx \\ &= x^5y^2 + x^6y^3 + x^4y^3 + g(y). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial F}{\partial y} &= 2x^5y + 3x^6y^2 + 3x^4y^2 + g'(y) \\ &= 2x^5y + 3x^6y^2 + 3x^4y^2 \qquad \Rightarrow \qquad g(y) = 0, \end{aligned}$$

and a general solution to the given equation is

$$x^5y^2 + x^6y^3 + x^4y^3 = C.$$

Section 2.5 problems

Problem 10 solution:

Since

$$\frac{\partial M}{\partial y}=1, \quad \frac{\partial N}{\partial x}=-1, \quad \text{and} \quad \frac{M_y-N_x}{N}=\frac{2}{-x},$$

the equation has an integrating factor

$$\mu(x)=\exp\left[\int\left(-\frac{2}{x}\right)dx\right]=\exp(-2\ln x)=x^{-2}.$$

Therefore, the equation

$$x^{-2}\left[\left(x^4-x+y\right)dx-xdy\right]=\left(x^2-x^{-1}+x^{-2}y\right)dx-x^{-1}dy=0$$

is exact. Therefore,

$$\begin{aligned}
&F(x,y)=\int\left(-x^{-1}\right)dy=-x^{-1}y+h(x),\\
&\frac{\partial F}{\partial x}=x^{-2}y+h'(x)=x^2-x^{-1}+x^{-2}y\\
&\Rightarrow \qquad h'(x)=x^2-x^{-1} \qquad \Rightarrow \qquad h(x)=\frac{x^3}{3}-\ln|x|\\
&\Rightarrow \qquad -\frac{y}{x}+\frac{x^3}{3}-\ln|x|=C \qquad \Rightarrow \qquad y=\frac{x^4}{3}-x\ln|x|-Cx.
\end{aligned}$$

Together with the lost solution, $x\equiv 0$, this gives a general solution to the problem.

Problem 14 solution:

Multiplying the given equation by $x^n y^m$ yields

$$\left(12x^n y^m+5x^{n+1}y^{m+1}\right)dx+\left(6x^{n+1}y^{m-1}+3x^{n+2}y^m\right)dy=0.$$

Therefore,

$$\begin{aligned}
\frac{\partial M}{\partial y}&=12mx^n y^{m-1}+5(m+1)x^{n+1}y^m,\\
\frac{\partial N}{\partial x}&=6(n+1)x^n y^{m-1}+3(n+2)x^{n+1}y^m.
\end{aligned}$$

Matching the coefficients, we get a system

$$\begin{cases} 12m = 6(n+1) \\ 5(m+1) = 3(n+2) \end{cases}$$

to determine n and m. This system has the solution $n = 3$, $m = 2$. Thus, the given equation multiplied by x^3y^2, that is,

$$\left(12x^3y^2 + 5x^4y^3\right) dx + \left(6x^4y + 3x^5y^2\right) dy = 0,$$

is exact. We compute

$$\begin{aligned} F(x,y) &= \int \left(12x^3y^2 + 5x^4y^3\right) dx = 3x^4y^2 + x^5y^3 + g(y), \\ \frac{\partial F}{\partial y} &= 6x^4y + 3x^5y^2 + g'(y) = 6x^4y + 3x^5y^2 \\ \Rightarrow \quad & g'(y) = 0 \quad \Rightarrow \quad g(y) = 0, \end{aligned}$$

and so $3x^4y^2 + x^5y^3 = C$ is a general solution to the given equation.

Problem 20 solution:

For the equation

$$e^{\int P(x)dx}\left[P(x)y - Q(x)\right] dx + e^{\int P(x)dx} dy = 0,$$

we compute

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(e^{\int P(x)dx}\left[P(x)y - Q(x)\right]\right) = e^{\int P(x)dx} P(x), \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(e^{\int P(x)dx}\right) = e^{\int P(x)dx} \frac{d}{dx}\left(\int P(x)dx\right) = e^{\int P(x)dx} P(x). \end{aligned}$$

Therefore, $\partial M/\partial y = \partial N/\partial x$, and the equation is exact.